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INVESTIGATION OF THIRD-ORDER CONTACTOR CONTROL SYSTEMS WITH TWO COMPLEX POLES WITHOUT ZEROS

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SUMMARY

The present paper reports an investigation of third-order systems under contactor control. The transfer function of the uncontrolled system has one real and two conjugate complex poles and may have zeros. In general, it is assumed that all poles have negative real parts; however, cases with positive real parts of the complex poles are included. The switching function depends on error and error derivatives, the error being the difference between the input and output of the system. The major portion of this report is devoted to systems without zeros in the transfer function; however, systems with zeros in the transfer function are discussed briefly.

It is shown that for an initial error (or a step input) the coefficients of a linear switching function $F = e + k_1 e' + k_2 e''$ (where k_1 and k_2 are coefficients and e, e', and e'' are the error and error derivatives, respectively) can be chosen in such a way that the system trajectory becomes optimum, that is, the origin (or the step height) can be reached in minimum time without chatter. Obviously, this statement is valid only for an ideal system with perfect relays and no transport delays.

The dependence of the coefficients of the error derivatives k_1 and k_2 on initial error or step height for fixed system parameters has been studied theoretically (behavior of the system trajectory in the phase space) and with the help of an analog computer. Results from both methods agree well.

In addition, the dependence of k_1 and k_2 on the system parameters has been considered. It appears that average switching-function coefficients assure good or quasi-optimum control for systems with varying parameters as long as the uncontrolled systems are stable.

Unstable systems can be satisfactorily controlled; however, one should not attempt to use the just-mentioned average k_1 and k_2 values.

The response of controlled systems to time-varying inputs has been studied also. In this case a small steady-state error occurs when the control coefficients are chosen in such a way that the relay is working in the chatter region.

In the last section of the paper, first studies of systems with zeros in the transfer function are described. A linear switching function of the type $F = e + k_1 e' + k_2 e''$ is bound to become discontinuous through the influence of the zeros in the transfer function of the uncontrolled system. If this type of switching function is still used, a new type of chatter occurs in addition to the well-known after-end-point chatter. The general third-order problem is still being studied.

INTRODUCTION

The investigation of contactor control systems has been confined to particular problems for quite some time. The nonlinearity of these systems for the most part prevents obtaining general results which can be exploited immediately for practical design purposes. One therefore can follow up definite trends in studying such systems.

Since the difficulty of theoretical treatment depends strongly on the poles of the transfer function of an uncontrolled system, a final division of problems is given by separating the systems into those with real poles only, complex poles only, or real and complex poles. There are a number of interesting theorems established for systems with real poles; unfortunately, however, many of the basic dynamic systems, especially aircraft and missiles, represent systems with complex or complex and real poles. As long as one is satisfied with simplifying the description of such a system to the degree that a transfer function of second order is obtained, the mathematical treatment is rather simple. A number of papers have been devoted to this problem. Switching depending on a linear combination of the sensed deviation and deviation derivative and the so-called optimum switching have been investigated for transfer functions with complex poles and without zeros. Reference 1 reports an investigation of the influence of zeros. Particular third-order systems with two complex and one real root and no zeros have been preliminarily investigated and reported in references 2 and 3.

The present report is devoted to a more complete investigation of the third-order system with two complex poles without zeros and a preliminary investigation of the influences of zeros. In general, so-called linear switching is investigated, which is a switching depending upon a linear combination of the sensed variables. In addition, the possibility of obtaining optimum switching by using linear switching functions is investigated. Optimum switching for a zero-seeking system, for instance, means reaching zero with a limited number of switchings and without chatter.

Since many of the systems to be controlled are systems with varying parameters (influence of Mach numbers in aircrafts, for example), the design of the contactor control should be such that the control mechanism is efficient even if these parameters change through a given range.

The theoretical approach to the problem is based on a phase-space representation of the motion, and theoretical results are confirmed and extended by analog-computer studies.

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SYMBOLS

A_1,A_2 constants in equation (A3)

$\bar{a}_0,\bar{a}_1,\bar{a}_2,\bar{a}_3,\bar{b}_0,\bar{b}_1,\bar{b}_2$ coefficients in differential equation of controlled system

b_1,b_2 coefficients in equation (39), $\bar{b}_1/\bar{b}_0$ and $\bar{b}_2/\bar{b}_0$, respectively

C,C_1,C_2,C_{1m},C_{2m} constants

D damping coefficient

D* damping coefficient of after-end-point motion

d coordinate in phase space; see equation (32)

e error, $x - y$

E_1,E_2,E_3 phase-space coordinates

F switching function

F^* modified switching function

k_1, k_2 coefficients of error derivatives in switching function

K^* defines angle between switching plane and E_1E_2 plane; see equations (33) and (34)

l_1, l_2, l_3 coefficients of switching plane in $E_1E_2E_3$ phase space

M^* constant

$N, \bar{N}$ amplitude of contactor output; see equation (6)

P differential operator, $d/d\tau$

x input variable

x_0 height of step input; also used for amplitude of sinusoidal input

y output variable

$Y(P)$ Laplace's transform of output variable

β constant factor in equation (A5)

γ real pole of uncontrolled system

$\delta, \bar{\delta}$ output of contactor, $\delta = \bar{\delta} \dfrac{\bar{b}_0}{\bar{a}_3}$

$\delta^*(P)$ Laplace's transform of contactor output

ϵ phase constant

ϵ_m phase constant in mth interval

λ_1, λ_2 roots of differential equation for after-end-point motion

$\nu = \sqrt{1 - D^2}$

$\sigma = \cos^{-1}(-D)$

$\sigma^* = \cos^{-1}(-D^*)$

τ	time variable
τ^*	time variable; see equation (A5)
τ_m	time variable in mth interval; $\tau_m = 0$ at start of interval
τ_i	dummy time variable
ω	angular frequency of sinusoidal input
$\Delta_1(\ldots)$	abrupt change of a variable at switching instant
$\Delta_2(\ldots)$	impulse of variable at switching instant
$\Delta^*(\ldots)$	combination of discontinuity and impulse; see equation (44)

Subscripts:

av	average values
m	mth interval
0	initial values
$(\ )', (\ )'', (\ )'''$	time derivatives
sgn( )	algebraic sign of a real quantity; $\mathrm{sgn}\, f = f/\lvert f\rvert$

BEHAVIOR OF THIRD-ORDER ZERO-SEEKING CONTACTOR CONTROL SYSTEMS IN PHASE SPACE

In the present section the equation of motion of a general third-order system is presented and the motion in phase space is described and discussed.

Equations of Third-Order System

The system treated in this report is of interest, for example, in studies of the longitudinal motion of an aircraft under certain simplifying assumptions. The equation of the system is given by

$$\bar{a}_3 y''' + \bar{a}_2 y'' + \bar{a}_1 y' + \bar{a}_0 y = \bar{b}_2 \bar{\delta}'' + \bar{b}_1 \bar{\delta}' + \bar{b}_0 \bar{\delta} \qquad (1)$$

where y and $\bar{\delta}$ are the output variable and the output of the contactor, the primes indicate derivatives with respect to time, and the coefficients $\bar{a}_n$ and $\bar{b}_n$ are constants. In Laplace's transform, equation (1) is expressed as follows:

$$Y(P) = \frac{\bar{b}_2 P^2 + \bar{b}_1 P + \bar{b}_0}{\bar{a}_3 P^3 + \bar{a}_2 P^2 + \bar{a}_1 P + \bar{a}_0} \bar{\delta}^*(P) \tag{2}$$

(The Laplace transform can be applied to equation (1) in an interval between two switchings. In such an interval the motion is governed by a linear differential equation.)

There have been a great number of papers written about contactor control systems, but most of them discuss the systems with real poles and little has been written for the case with a real pole and two conjugate poles. In this section the latter system is considered for the case $\bar{b}_2 = \bar{b}_1 = 0$.

The denominator of equation (2) can be written as a product of a second-order and a first-order term. Therefore, equation (2) with $\bar{b}_2 = \bar{b}_1 = 0$ yields

$$Y(P) = \frac{\left(\bar{b}_0/\bar{a}_3\right)}{(P + \gamma)(P^2 + 2DP + 1)} \bar{\delta}^*(P) \tag{3}$$

where γ is a real pole and $D < 1$ because, as indicated in the introduction, only third-order systems with two complex roots are considered in this report.

Equation (3) can be rewritten as

$$Y(P) = \frac{\left(\bar{b}_0/\bar{a}_3\right)}{P^3 + (\gamma + 2D)P^2 + (2D\gamma + 1)P + \gamma} \bar{\delta}^*(P) \tag{4}$$

and the original equation (1) takes the form

$$y''' + (\gamma + 2D)y'' + (2D\gamma + 1)y' + \gamma y = \bar{\delta}\,\frac{\bar{b}_0}{\bar{a}_3} = \delta \tag{5}$$

where $(\gamma + 2D) = \left(\bar{a}_2/\bar{a}_3\right)$, $(2D\gamma + 1) = \left(\bar{a}_1/\bar{a}_3\right)$, and $\gamma = \bar{a}_0/\bar{a}_3$. The variable $\bar{\delta}$ is the output of a contactor and takes only two discrete values $\bar{N}$ and $(-\bar{N})$ according to the sign of the input. That means

$$\delta = (\bar{N} \text{ sgn } F)\frac{\bar{b}_0}{\bar{a}_3} = N \text{ sgn } F \tag{6}$$

where F is a function of the error e and its derivatives

$$F = F(e,e',e'') \tag{7}$$

The error e is given by

$$e = x - y \tag{8}$$

The type of switching function considered is a linear combination of the error and its derivatives

$$F = e + k_1 e' + k_2 e'' \tag{9}$$

The block diagram of the contactor control system described above is shown in figure 1.

Substitution of equation (8) into equation (5) gives an equation describing the system in terms of error and its derivatives.

$$\begin{aligned} &e''' + (\gamma + 2D)e'' + (2D\gamma + 1)e' + \gamma e \\ &\qquad = x''' + (\gamma + 2D)x'' + (2D\gamma + 1)x' + \gamma x - N \text{ sgn } F \end{aligned} \tag{10}$$

It should be mentioned that the variables e, e', and e'' are continuous at the switching points. However, equations (5) and (10) show clearly that e''' is discontinuous at the switching points, which means that the derivative of the chosen switching function $dF/d\tau$ is discontinuous at the switching points. This indicates the possibility of reaching "end points" in an ideal system as described in reference 4 on page 32.

For a zero-seeking system

$$x = x' = x'' = x''' = 0 \tag{11}$$

and equation (10) becomes

$$e''' + (\gamma + 2D)e'' + (2D\gamma + 1)e' + \gamma e = -N \text{ sgn } F \tag{12}$$

Equation (12) can be solved for the mth interval between switchings since $\delta = N \operatorname{sgn} F$ is a constant.

The error and its derivatives are given by

$$e(\tau) = C_{1m} e^{-D\tau_m} \cos(\nu\tau_m + \epsilon_m) + C_{2m} e^{-\gamma\tau_m} - (N/\gamma) \operatorname{sgn} F \tag{13}$$

$$e'(\tau) = C_{1m} e^{-D\tau_m} \cos(\nu\tau_m + \epsilon_m + \sigma) - C_{2m} \gamma e^{-\gamma\tau_m} \tag{14}$$

$$e''(\tau) = C_{1m} e^{-D\tau_m} \cos(\nu\tau_m + \epsilon_m + 2\sigma) + C_{2m} \gamma^2 e^{-\gamma\tau_m} \tag{15}$$

where $D = -\cos \sigma$, $\nu = \sqrt{1 - D^2} = \sin \sigma$, and C_{1m}, C_{2m}, and ϵ_m are constants determined by the initial conditions for the interval.

Since switching is initiated by the condition

$$F = 0 = e + k_1 e' + k_2 e'' \tag{16}$$

the time of switching is determined by the solution of a transcendental equation. However, if the motion is described by a trajectory in phase space (coordinates e, e', and e'') all switching points lie in a plane.

The description of the motion by phase-space variables has proven to be very convenient for a second-order system. In that case the motion can be described in a phase plane, and the switching points lie on a straight line if a linear switching function is used. The representation of a third-order system in phase space is still very convenient when certain linear combinations of e, e', and e'' are used as phase variables.

Introduction of New Phase Coordinates

The suggested set of transformed coordinates is

$$E_1 = \gamma e' + e'' \tag{17}$$

$$E_2 = \gamma e + e' \tag{18}$$

$$E_3 = e'' + 2De' + e \tag{19}$$

As shown in figure 2, the phase space is constructed with a set of coordinates E_1 and E_2 which are skewed with respect to each other and E_3 which is perpendicular to the E_1E_2 plane.

Substitution of the error and its derivatives into equations (17), (18), and (19) gives

$$E_1 = C_{1m}e^{-D\tau_m}\left[\gamma \cos\left(\nu\tau_m + \epsilon_m + \sigma\right) + \cos\left(\nu\tau_m + \epsilon_m + 2\sigma\right)\right] \tag{20}$$

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$$E_2 = C_{1m}e^{-D\tau_m}\left[\gamma \cos\left(\nu\tau_m + \epsilon_m\right) + \cos\left(\nu\tau_m + \epsilon_m + \sigma\right)\right] - N \operatorname{sgn} F \tag{21}$$

$$E_3 = C_{2m}\left(\gamma^2 - 2D\gamma + 1\right)e^{-\gamma\tau_m} - (N/\gamma)\operatorname{sgn} F \tag{22}$$

The equation of the switching plane in this new phase space is given by

$$F = e + k_1e' + k_2e''$$

$$= \frac{1 - k_1\gamma + k_2(2\gamma D - 1)}{-\gamma^2 + 2D\gamma - 1} E_1 + \frac{(2D - \gamma) - k_1 + k_2\gamma}{-\gamma^2 + 2D\gamma - 1} E_2 + \frac{-1 + k_1\gamma - k_2\gamma^2}{-\gamma^2 + 2D\gamma - 1} E_3$$

or

$$F = l_1E_1 + l_2E_2 + l_3E_3 \tag{23}$$

where

$$l_1 = \frac{1 - k_1\gamma + k_2(2\gamma D - 1)}{-\gamma^2 + 2D\gamma - 1} \tag{24}$$

$$l_2 = \frac{(2D - \gamma) - k_1 + k_2\gamma}{-\gamma^2 + 2D\gamma - 1} \tag{25}$$

$$l_3 = \frac{-1 + k_1\gamma - k_2\gamma^2}{-\gamma^2 + 2D\gamma - 1} \tag{26}$$

A special case that should be mentioned here is that of a system with $\gamma = 0$, which is practically the same as the so-called second-order velocity-control system of reference 4. The system equation is given by

$$e''' + 2De'' + e' = -N \operatorname{sgn} F \tag{27}$$

The new phase variables are given by

$$\left.\begin{aligned} E_1 &= e'' \\ E_2 &= e' \\ E_3 &= e'' + 2De' + e \end{aligned}\right\} \tag{28}$$

and as functions of time by

$$E_1 = C_{1m}e^{-D\tau_m}\cos\left(\nu\tau_m + \epsilon_m + 2\sigma\right) \tag{29}$$

$$E_2 = C_{1m}e^{-D\tau_m}\cos\left(\nu\tau_m + \epsilon_m + \sigma\right) - N \operatorname{sgn} F \tag{30}$$

$$E_3 = C_{2m} - N \operatorname{sgn} F\tau_m \tag{31}$$

These solutions can be obtained from equations (20) to (22) by a limiting process; however, it is simpler to derive them directly from equation (27).

The system behavior can be best observed by plotting the projections of the trajectory in the E_1E_2 plane and in the E_3d plane, which is perpendicular to the E_1E_2 plane and also to the switching plane (figs. 3(a) and 3(b)). The new coordinate d is a linear combination of E_1 and E_2. In the E_3d plane the switching plane appears as a straight line. Additional diagrams showing E_3 as a function of time (figs. 3(c) and 3(d)) proved to be very useful.

Projection of trajectory into E_1E_2 plane.- As can be seen from equations (20) and (21) or (29) and (30), the projection of the trajectory into the E_1E_2 plane is composed of portions of logarithmic spirals around two centers N and $(-N)$ (see also ref. 4, p. 24). Which center an arc belongs to depends on the sign of the switching function F. An example of such a projection on the E_1E_2 plane is given in figure 3(a), where P_1 and P_2 are switching points. The time is measured by the angle subtended by an arc around the center.

Projection of trajectory into E_3d plane.- The switching plane $F = 0$ and the E_1E_2 plane intersect in a straight line. The coordinate axis d (figs. 2 and 3(a)) is perpendicular to this intersection line. Figure 3(b) shows the projection of a trajectory onto the E_3d plane.

The coordinate d depends on E_1 and E_2 in the following way:

$$d = -\sqrt{1 - D^2}\,\frac{l_1E_1 + l_2E_2}{\sqrt{l_1^2 - 2l_1l_2D + l_2^2}} \tag{32}$$

The derivation of the above expression is quite similar to the derivation given for d_1 in reference 4, page 57.

The intersection of the switching plane with the E_3d plane is given by

$$E_3 = K^*d \tag{33}$$

with

$$K^* = \frac{\sqrt{l_1^2 - 2l_1l_2D + l_2^2}}{\sqrt{1 - D^2}}\,\frac{1}{l_3} \tag{34}$$

Diagram of E_3 as function of τ.- It is clear from equation (22) that for positive values of γ the phase variable E_3 decays exponentially toward N or $(-N)$ depending on the sign of the switching function as shown in figure 3(c). For systems with $\gamma = 0$, equation (31) yields a linear dependence of E_3 on τ (see fig. 3(d)). Correspondence between E_3 and E_1 or E_2 is given through τ which can be measured by the angle subtended by the different arcs in the E_1E_2 plane. These relations are demonstrated by the following examples.

Examples of System Behavior in Phase Space

Three examples will be presented for familiarizing the reader with the behavior of the third-order system without zeros and with a linear switching function.

Example (1) is a system with $\gamma \neq 0$ described by:

$$D = 0.2$$

$$\gamma = 1$$

$$N = 2$$

The switching coefficients are chosen as

$$k_1 = 1$$

$$k_2 = 1.2$$

The motion starts with the values

$$E_{10} = 0$$

$$E_{20} = 5.0$$

$$E_{30} = 5.0$$

which correspond to

$$e_0 = 5.0$$

$$e_0' = 0$$

$$e_0'' = 0$$

In figure 4(a) the motion is shown by projections of its trajectory into the E_1E_2 plane and into the E_3d plane. The E_1E_2 plane is rotated for ease of plotting. The diagram $E_3(\tau)$ is added to demonstrate the decay of E_3 with time. This diagram facilitates the design of the E_3d projection of the trajectory.

The motion starts at P_0, then proceeds around the spiral center (-N) until it hits the switching plane at P_1 as is shown in the E_3d plane. After passing P_1 the motion proceeds around the other center N toward P_2. At P_2 it is obvious that the motion cannot proceed any farther because the switching command cannot be executed. If one tried to switch, one would arrive at a situation which would demand a motion around the wrong center. An end point for the motion of an ideal system is P_2; however, there are no ideal systems, and the ever-present time delay in the contactor allows the motion to proceed toward the origin in the switching plane as indicated in the E_3d plane. Details for the after-end-point chatter motion in this third-order system are discussed in the appendix. The after-end-point chatter motion for second-order systems is discussed in detail in reference 3.

Example (2) is a system with $\gamma = 0$. The essential data are

$$D = 0.2$$

$$\gamma = 0$$

$$N = 2$$

The coefficients of the switching function are

$$k_1 = 1$$

$$k_2 = 2$$

The initial conditions are given by

$$E_{10} = 0$$

$$E_{20} = 5.0$$

$$E_{30} = 3.0$$

which correspond to

$$e_0 = 1.0$$

$$e_0' = 5.0$$

$$e_0'' = 0$$

As shown in figure 4(b) the motion proceeds similarly to that of the previous example except that E_3 changes linearly with τ. At P_2 an end point is reached, and the average motion proceeds according to the equation of the switching plane

$$e_{av} + k_1 e_{av}' + k_2 e_{av}'' = 0$$

toward the coordinate origin. That means

$$e_{av} = Ce^{-0.25\tau}\cos(0.66\tau + \epsilon)$$

where C and ϵ are determined by e and e' at the end point.

Example (3) is chosen to show what happens when the coefficients of the switching function F are not properly selected. The system data are

$$D = 0.2$$

$$\gamma = 0$$

$$N = 2.0$$

The coefficients of the switching function are

$$k_1 = 0.8$$

$$k_2 = 0.2$$

The initial values are

$$E_{10} = 0$$

$$E_{20} = 5.0$$

$$E_{30} = 0$$

or

$$e_0 = -2.0$$

$$e_0' = 5.0$$

$$e_0'' = 0$$

Figure 4(c) shows that the projection of the trajectory into the E_1E_2 plane tends strongly toward a limit cycle, and the projection into the E_3d plane does the same, however, somewhat more slowly. The final state is an oscillation of the system with considerable amplitude.

A poor selection of k_1 and k_2 may cause a system to oscillate or even diverge. The ratio of the initial values to N has a great influence on the system behavior as can be concluded easily from studying the trajectories.

OPTIMUM RESPONSE OF THIRD-ORDER CONTACTOR CONTROL SYSTEMS WITH LINEAR SWITCHING FUNCTIONS

In the previous section new phase coordinates were introduced, and the behavior of systems was studied by observing their trajectories in the phase space. The purpose of this section is to obtain optimum response in systems with linear switching functions. Optimum response for zero-seeking systems and for step-function input are discussed in detail. The results obtained in this section are used in the next section for studying quasi-optimum response.

Optimum Response of Zero-Seeking Systems

Zero-seeking systems are systems designed to reduce an initial error to zero. Their response will be called optimum if the error and its derivatives are reduced to zero in minimum time. A switching function F which gives optimum response is called an optimum switching function. For second-order systems $F = 0$ represents a curve in the phase plane; for third-order systems $F = 0$ is given by a surface in the phase space. The optimum switching curve for second-order contactor control systems has been extensively investigated by Bushaw (ref. 5) and the optimum surface for third-order systems is under investigation by Yin (ref. 6).

The systems discussed in the present paper have linear switching functions. It is important to answer whether optimum response for a given disturbance is realizable with this arrangement. The answer is affirmative if the number of switchings is limited to only two. Suppose the system starts from an initial point, switches twice on the optimum surface, and goes into the origin. Now if the orientation and the slope of the switching plane is chosen so that those two switching points on the surface and the origin are on this plane, optimum response can be realized by a linear switching function. However, it is obvious that the optimum switching plane depends on the initial point of the trajectory.

It is quite easy to construct examples of the optimum trajectory in phase space. It is convenient to plot the trajectory backward, starting from the origin on a zero trajectory, and to proceed in negative time. Examples are given in figure 5 for the cases of $\gamma = 0$ and $\gamma = 1$. After the second switching point P_1, any point on the final spiral could be chosen as the initial point. Once switching points P_1 and P_2 are chosen, it is possible to determine a plane which passes through these points and the origin. The coefficients k_1 and k_2 in equation (16) determine the switching plane and can be obtained by solving a system of two linear equations.

Optimum Response for Step Input

It is a common technique to investigate the response of linear systems through the response to a step input. The systems treated in this report are nonlinear, and hence the law of superposition does not hold. However, it will be valuable to have the knowledge of step-function response because it shows the response to a fast-changing input.

Referring to equation (10), it is noticed that the right-hand side of the equation contains terms of input and its derivatives. For step input, the following relations hold:

$$x = x_0 = \text{Constant} \tag{35}$$

and

$$x' = x'' = x''' = 0 \tag{36}$$

Thus, equation (10) becomes

$$e''' + (\gamma + 2D)e'' + (2D\gamma + 1)e' + \gamma e = \gamma x_0 - N \operatorname{sgn} F \tag{37}$$

As is seen, equation (37) has a term γx_0 on the right-hand side. Previous investigations, especially Lindberg's, have shown that stability of such systems can be expected only for

$$|\gamma x_0| < |N| \tag{38}$$

One can understand this immediately. The existence of γx_0 in equation (37) means that the spiral centers N and (-N) are shifted up or down by γx_0. The system fails if $|\gamma x_0|$ is bigger than N.

This shifting effect can be removed by feeding the input x into the controlled system directly as shown in figure 6. This feeding of the input removes the shifting term γx_0 and makes the system capable of handling any size of step input theoretically. The forward feeding of x has been applied to all systems described in the present paper, although it has to be borne in mind that the system parameters are not always quite constant. Thus, it is not advisable to design a system such that γx is much greater than N for the sake of the stability of the system. For systems having γ equal to zero, this forward feeding of x is not required.

Graphical Determination of Optimum Switching Plane for Step Input

For systems with forward feeding of x, the response to step inputs is the same as for zero-seeking systems with initial error but no initial error derivative and no initial error acceleration. Therefore, trajectories can be traced backward as described earlier. The initial point is given by $e_0 \neq 0$, $e_0' = 0$, and $e_0'' = 0$ or $E_{10} = 0$, $E_{20} = \gamma e_0$, and $E_{30} = e_0$. This means that for $\gamma = 0$ the initial values E_{10} and E_{20} are both zero, or that in the E_1E_2 plane the initial point and the point which indicates the end of motion are coinciding and are lying in the coordinate origin. Figure 7(a) shows projections of trajectories in the E_1E_2 plane for a system with $\gamma = 0$. Figure 7(b) gives the corresponding variation of E_3 with time. Notice that the plot is done backward in time. The trajectory $P_0P_1P_2O$ belongs to a rather low step input. The larger input corresponds to a larger loop. At first glance the trajectory $P_0\overline{P}_1\overline{P}_2O$ seems to indicate a limit to the height of the step input which still can be followed up with two switchings; however, this is not the case. Consider figure 7(c) which shows some interesting trajectories. It is obvious that the trajectory portion P_0P_1 can wind around $(-N_1)$ many times and still be allowed to continue toward the zero trajectory. In principle, one could have infinite E_3, which is equivalent to leaving the P_1 projection at $(-N_1)$. This fixes the arcs P_1P_2 and P_2O. For lesser, but rather large, heights the switching point P_1 would lie very close to $(-N_1)$ and, therefore, one concludes that for rather large heights the location of the switching points P_1 and P_2 in the E_1E_2 plane changes very little and their E_3 coordinates are practically fixed. This condition is equivalent to having k_1 and k_2 tend toward limit values, because for $\gamma = 0$ the values l_1, l_2, and l_3 are given by $l_1 = k_2 - 1$, $l_2 = k_1 - 2D$, and $l_3 = 1$ and l_1 and l_2 depend on the location of P_1 and P_2.

The case $D = 0$ demands special consideration. In this case the trajectory projection in the E_1E_2 plane degenerates to circles.

Figure 7(d) shows that $\bar{P}_1$ and $\bar{P}_2$ coincide. Repeated winding around (-N) allows large step inputs without any switching.

These last considerations are very interesting for determining limit values for k_1 and k_2; however, in practice one should not try to make the ratio x_0/N too large, as was mentioned earlier.

In figure 7(e) trajectories are shown for systems with $\gamma \neq 0$. In this case the initial point lies on the E_2 axis. Switching points have to be chosen such that $E_{20} = \gamma E_{30}$. This means that the determination of an optimum path in the E_1E_2 plane can only be done by a trial-and-error method with the diagram of E_3 against τ as essential help.

W138

Simulation of Systems on Analog Computer

Details of the simulation on the analog computer are given in figure 8. The wiring is complete in the sense that variable inputs can be studied. Since E_1, E_2, and E_3 are only auxiliary variables, it is satisfactory to have e, e', and e"; however, for comparison with the theoretical work it is desirable to see E_3 against τ. Therefore, a special arrangement for obtaining $E_3 = e + 2De' + e''$ is made.

It is possible to determine an optimum set of k_1 and k_2 on the computer by observing the responses on an oscilloscope and by diagrams drawn by a pen recorder. When projections on the E_1E_2 plane are watched in the oscilloscope, $E_3(\tau)$ is also observed; the coefficients k_1 and k_2 are varied until optimum response is obtained.

In figures 9(a) and 9(b) the optimum values of k_1 and k_2 for different heights of input steps are given for systems with $\gamma = 0$ and $\gamma = 1$, respectively, and $D = 0.2$. Also, values obtained by the graphical method are given in the same figures. It is apparent that the values taken by the two methods lie close together, proving the accuracy of simulation. From now on all the data to be taken will be obtained by simulation.

The height of the input step is limited to a region $0 < x_0/N < 4$ because, as mentioned earlier, it is not recommended that systems be designed for a higher level input.

For the reader who compares these results with those in reference 3, the following remarks will be helpful. The k coefficients $(k_1$ and $k_2)$ appearing in figure 28 of reference 3 are defined for a system in which a different time scale is used. The values of k have to be normalized to see the correspondence. Equation (129) of reference 3 states that

$$e''' + 2\zeta\Omega e'' + \Omega^2 e' = \Omega^2 \overline{V}_{ra}\phi$$

where ζ is the damping ratio, Ω is the natural frequency of the undamped third-order system, and $\overline{V}_{ra}$ is the runaway velocity of the third-order controlled process and ϕ is the switching function. Upon using the following set of transformations (subscript L refers to the notations used in ref. 3)

$$\Omega\tau_L = \tau$$

$$\zeta_L = D$$

$$\frac{\overline{V}_{ra}}{\Omega} = N$$

in the above equation one can derive equation (27). It is obvious that $k_{1L}\Omega = k_1$ and $k_{2L}\Omega^2 = k_2$.

Dependence of Coefficients k_1 and k_2 on Height of Step Input for Various System Characteristics D and γ

It is known that the equation of motion of the uncontrolled airframe changes its constants D and γ as the speed of flight changes. The value of D could even be negative for particular speeds. Since it is one of the purposes of this investigation to design a stable control system with a linear switching function, the dependence of optimum sets of k_1 and k_2 values upon the frame constants D and γ for various step inputs have been investigated.

Figure 9(c) shows the variation of k_1 and k_2 with variation of step input with D as parameter for systems with $\gamma = 0$. As mentioned earlier, k_1 and k_2 tend toward limit values with increasing x_0/N ratio for positive values of D (approaches are oscillatory, see ref. 3). However, a similar consideration shows that for negative values of D both constants will tend toward infinity with increasing step height.

Figure 9(d) shows the variation of k_1 and k_2 in the same way for systems with $\gamma = 1$. It should be noted that larger step heights demand higher values of k_1 and k_2 with decreasing D. There seems to be for each D a particular step height at which a steep increase for the k is starting. Thus far no simple criterion for this breakaway step height has been found.

In figure 9(e) the influence of different values of γ for fixed $D = 0.2$ on the magnitudes of optimum k_1 and k_2 is shown. From figure 3(c) it could be expected that the value of γ has a strong influence on k_1 and k_2. Since γ determines the slope of the $E_3(\tau)$ curve, one might consider it understandable that an increase of γ acts as an increase of N. This agrees with the appearance of the curves for $\gamma = 2$ when compared with those for $\gamma = 1$ in figure 9(e).

It has been shown in figures 9(c), 9(d), and 9(e) that the variations of k_1 and k_2 are quite large for higher levels of input, especially for the negatively damped case. It can be concluded that the use of a fixed set of values of k should be limited to a lower level input.

QUASI-OPTIMUM RESPONSE OF THIRD-ORDER CONTACTOR CONTROL SYSTEMS TO STEP INPUTS

The purpose of the present section is to study the response of systems with a fixed set of k_1 and k_2 selected from the data from the previous section, in which the variations of optimum switching coefficients k_1 and k_2 for step inputs were investigated. The constants D and γ of the systems under control are varied.

A third-order contactor control system with a fixed switching plane could give an optimum response for a particular height of step or a particular initial disturbance. For steps or initial points close to that step or initial point the response of the system will be quite close to optimum. Such response will be called quasi-optimum response. An adaptive system should give quasi-optimum response for various heights of step input.

Response of Systems With Fixed D for Various Heights of Step Input

As is seen in figure 9(c), variations of k_1 and k_2 are rather small for $D = 0.2$. In such a case a medium set of k_1 and k_2 is expected to give good response for various heights of steps. The set used here is

$$k_1 = 1.00$$

$$k_2 = 0.37$$

The results are found to be fairly good for input amplitudes

$$\frac{20}{25} < \frac{x_0}{N} < \frac{80}{25}$$

In figures 10(a) and 10(b) responses are given for a system with $D = 0.2$ and $\gamma = 0$. Error e and the phase variable $E_3 = e + 2De' + e''$ are plotted for two step inputs $x_0/N = 20/25$ and $x_0/N = 80/25$. In both cases there are two essential switchings and a slight chatter toward the end of the motion.

The next example is an interesting test case because the uncontrolled system is negatively damped. The selection of the values of k is essentially determined by the stability of the controlled system. Figures 10(c) and 10(d) show responses for a system with $D = -0.2$ and $\gamma = 0$ and coefficients $k_1 = 1.30$ and $k_2 = 0.93$. For $x_0/N = 40/25$ an end point is reached too fast, and for $x_0/N = 100/25$ the response is oscillatory. Larger values of k would diminish the oscillation, but the response would be slower and far from optimum.

The third test run is for systems with $\gamma \neq 0$. In figures 10(e) and 10(f) responses are shown for a system with $D = 0.2$, $\gamma = 1$, $k_1 = 1.70$, and $k_2 = 0.45$. There are essentially two switchings, reaching an end point with slight chatter. The responses are satisfactory when the wide range of step heights investigated and their influence on k_1 and k_2 (see fig. 9(b)) are considered.

It is shown in these examples that in order to obtain quasi-optimum response for various inputs the level of input should be smaller than the restoring force N.

Response of Systems With Varied D and Fixed Switching Coefficients to Step Inputs

As mentioned earlier, airframes change their coefficients with the speed of flight. Therefore, it is important to design control systems which operate satisfactorily under such conditions.

The controlled system must be stable at all times and must give good responses. These two requirements determine the choice of the switching plane; therefore, there will be a compromise in any case.

The first example is given in figures 10(g) and 10(h). Responses are recorded for a system with $\gamma = 0$ under two extreme conditions, $D = 0.4$ and $D = -0.2$. The values of k_1 and k_2 are chosen to be a little larger than the average of the optimum values of k_1 and k_2 for each case. The responses seem to be fairly good for both cases.

The second example (figs. 10(i) and 10(j)) is a similar test for a system with $\gamma = 1$. The response is a little worse than in the preceding example, as can be expected from figure 9(d). The variation of optimum values of k is bigger for this system.

RESPONSE OF QUASI-OPTIMUM SYSTEMS TO TIME-VARYING INPUT

The study of the response to step inputs was only a means to test systems in a simple way. It shows the reaction of systems to a very fast changing input. However, it is necessary to test the quasi-optimum systems for their responses to more general time-varying inputs.

An optimum system should follow the input with minimum instantaneous error at all times. This is a rather strict requirement, and an analytical treatment of this problem is very difficult. Contactor systems can follow a varying input best when working most of the time in the chatter region. This implies that frequencies of the input signal have to be lower than the chatter frequency, as discussed in reference 3.

The behavior of systems with fixed switching coefficients was studied for sinusoidal and for irregular inputs. Sinusoidal inputs have the advantage that their derivatives can be obtained without using differentiators. The term "irregular" input is used instead of random input because no instrumentation for producing a truly "random" input was at hand when these experiments were done.

Sinusoidal Input

A sinusoidal wave $x = x_0 \cos \omega\tau$ is generated on the same computer board as that used for other inputs and is used as input signal. The derivatives are obtained at the same time.

The k_1 and k_2 values are selected for the maximum value of $|x|$, that is, the amplitude of the cosine wave, because the system should be pulled into the input signal as fast as possible. The chatter frequency depends on the time delay in the contactor (ref. 3). In the present study time delay was minimized by using diodes as switching devices.

Figure 11 shows the results of studying responses to sinusoidal inputs for systems with $D = 0.2$ and $\gamma = 0$ and $\gamma = 1$. The maximum error at steady state increases slowly with the frequency of the input and jumps suddenly when the maximum local acceleration of the input becomes too big (breakdown frequency, see ref. 3). It is evident that the system with $\gamma \neq 0$ is inferior to that with $\gamma = 0$.

Figures 12(a) to 12(d) present the response of systems with varying parameter D. The real pole is fixed; $\gamma = 0$ in figures 12(a) and 12(b) and $\gamma = 1$ in figures 12(c) and 12(d). The steady-state error is amazingly small in both cases. It is worth noticing that the systems do not have any phase shift, which is inherent in linear systems. After a small transient the system follows the input signal closely, with a steady-state error entirely due to the chatter of the contactor.

Another example in which the parameter γ is varied is given in figures 12(e) to 12(g). The steady-state peak-to-peak error seems to increase with γ, which cannot be quite explained in the present stage of investigation.

Irregular Input

An irregular time-varying input was fed into the system. The irregular input is a kind of modified saw-tooth curve. Rather large errors have to be expected near the points of abrupt change of slope where the second derivative x'' is rather large and the imperfections of the differentiators become noticeable.

As seen from equation (10), the higher derivatives of x may reinforce or weaken the influence of the term $N \operatorname{sgn} F$, which is equivalent to a shift of the spiral centers in the E_1E_2 plane in a favorable or unfavorable way.

The test runs were performed with systems having varying parameters D and γ. These are

$$\gamma = 0, \quad N = 25, \quad D = -0.2,\ 0,\ 0.2,\ 0.4$$

$$\gamma = 1, \quad N = 25, \quad D = -0.2,\ 0,\ 0.2,\ 0.4$$

$$D = 0.2, \quad N = 25, \quad \gamma = 0,\ 1,\ 2$$

The switching coefficients $k_1 = 1.30$ and $k_2 = 0.30$ are used. The values of these coefficients are primarily chosen for the sake of the stability of the systems. The results are quite alike for all the systems, and some of them are shown in figure 13.

This random input should still be investigated in greater detail.

PRELIMINARY EXTENSION TO SYSTEMS WITH ZEROS

In the preceding sections it was assumed that the transfer function of the third-order system has two complex poles and one real pole, but no zeros. It is the purpose of this section to present some preliminary considerations on the systems with zeros. The phase-space coordinates E_1, E_2, and E_3 as defined in equations (17), (18), and (19) are used for a theoretical description of the motion of such general systems, and a linear switching function is assumed.

The third-order system with zeros is presented by equation (1). Following the considerations of the section "Equations of Third-Order Systems" this equation can be written in the following form:

$$y''' + (\gamma + 2D)y'' + (2D\gamma + 1)y' + \gamma y = b_2\delta'' + b_1\delta' + \delta \tag{39}$$

where

$$\delta = N \operatorname{sgn} F \tag{40}$$

and

$$F = e + k_1e' + k_2e'' \tag{9}$$

Equation (39) can be rewritten in a more convenient form by using e and x instead of y

$$e''' + (\gamma + 2D)e'' + (2D\gamma + 1)e' + \gamma e$$
$$= x''' + (\gamma + 2D)x'' + (2D\gamma + 1)x' + \gamma x - b_2\delta'' - b_1\delta' - \delta \qquad (41)$$

For a zero-seeking system input x and its derivatives are zero and equation (41) becomes simply

$$e''' + (\gamma + 2D)e'' + (2D\gamma + 1)e' + \gamma e = -b_2\delta'' - b_1\delta' - \delta \qquad (42)$$

Equation (42) has the same solutions as those described earlier for the mth interval between switchings since δ is a constant and its derivatives are zero in such an interval. But at the instant of switching of the contactor, the presence of the derivatives of δ in equation (42) becomes significant. The value of δ changes abruptly, which means that δ' is equivalent to an impulse and δ'', to a double impulse. Referring to equation (42) it should be noticed that to maintain the equality at the switching instant the derivative e''' must contain a double impulse, and e'' must have an impulse when e' is changing abruptly. The equation should be integrated over a short interval including the switching instant for finding these discontinuities. The process of this integration is well explained in reference 1. If the switching instant is called $\tau = 0$, the integration stretches from $\tau = -\Delta/2$ to $\Delta/2$, with Δ tending toward zero in the limiting process.

$$\lim_{\Delta\to 0} \int_{-\Delta/2}^{\Delta/2} \left[e''' + (\gamma + 2D)e'' + (2D\gamma + 1)e' + \gamma e\right]d\tau$$
$$= \lim_{\Delta\to 0} \int_{-\Delta/2}^{\Delta/2} \left(-b_2\delta'' - b_1\delta' - \delta\right)d\tau \qquad (43)$$

or

$$\Delta^{*}e'' + (\gamma + 2D)\Delta_1 e' = -b_2\Delta_2\delta' - b_1\Delta_1\delta \qquad (44)$$

where Δ_1 signifies a discontinuity, Δ_2 an impulse, and Δ^* a combination of both. A double integral shows the discontinuity in e' at the switching instant:

$$\lim_{\Delta\to 0} \int_{-\Delta/2}^{\Delta/2} \left\{ \int_{-\Delta/2}^{\tau} \left[e''' + (\gamma + 2D)e'' + (2D\gamma + 1)e' + \gamma e \right] d\tau_1 \right\} d\tau$$

$$= \lim_{\Delta\to 0} \int_{-\Delta/2}^{\Delta/2} \left[\int_{-\Delta/2}^{\tau} \left(-b_2\delta'' - b_1\delta' - \delta \right) d\tau_1 \right] d\tau$$

or

$$\Delta_1 e' = -b_2 \Delta_1 \delta \tag{45}$$

This shows the presence of a discontinuity in e'. Equation (45) is substituted into equation (44) to get the abrupt changes in e'':

$$\Delta^* e'' = \left[b_2(\gamma + 2D) - b_1 \right] \Delta_1 \delta - b_2 \Delta_2 \delta' \tag{46}$$

It is seen that the above expression contains an impulse δ'; but the important result is the discontinuity of e'' at switching, and the last term $b_2\Delta_2\delta'$ is not significant for the construction of the trajectory. The discontinuity in e'' at switching is given by

$$\Delta_1 e'' = \left[b_2(\gamma + 2D) - b_1 \right] \Delta_1 \delta \tag{47}$$

Therefore, the motion of the system in the error-phase space can be constructed as in the cases without zeros except that there is a discontinuity in e'' and e' at every switching instant.

In case the transfer function has only one zero, which means that $b_2 = 0$, there is still a discontinuity in e'', but no discontinuity in e' at switching.

The switching function $F = e + k_1 e' + k_2 e''$ has discontinuities of first and second order for the general case ($b_2 \neq 0$; $b_1 \neq 0$), and a first-order discontinuity only if $b_2 = 0$. The derivative $dF/d\tau$ is discontinuous at switchings, even if the value $k_2 = 0$ should be chosen. It means that end points and after-end-point chatter may occur even in this special case.

A detailed study of such systems is being carried out at this time but is not reported in the present paper. Here only some special features of the control of systems with zeros will be mentioned.

Figure 14 shows the behavior of F near switching points for the case $b_2 = 0$. Let a positive F in figure 14 approach zero. At zero a switching command is given, and immediately, for positive b_1 and k_2, F tries to jump back to a finite positive value. However, the point S_1^* can be reached only if at this time δ actually changes from N to $(-N)$. Since at S_1^* the value $F > 0$ exists, δ cannot be negative. In other words, every zero point of F reached on a regular trajectory proves to be an end point for an ideal system. If a delay in executing the switching command is assumed, the broken curve represents the behavior of F. The jumping F creates a new switching point whose command will be followed again with a delay. The necessary jump creates a third switching point, and so on. It becomes obvious that the first zeroing of F starts a chatter motion. The motion will be determined by the average value of δ in this chatter region. If this should be zero, uncontrolled chatter will occur. When $\delta_{av} = 0$, δ stays at N and at $(-N)$ for equal amounts of time. As soon as δ_{av} is no longer zero (that change occurs at about τ_1 in fig. 15), another type of chatter occurs. Actual examples (not included in this report) show that for $b_1 > 0$ and $k_2 > 0$ the motion starts on a regular trajectory. At the very first point where $F = 0$, uncontrolled chatter motion begins. Some time later, this changes to controlled chatter and finally changes again to uncontrolled chatter, which then leads to a tiny limit cycle around the origin of the phase space. The dimensions of this limit cycle depend entirely on the imperfections of the system (time delay, threshold, dead-zone).

In the case of $b_1 > 0$, $k_2 < 0$ the jump occurs in the other direction (see fig. 16), and the motion proceeds without chatter.

The cases with $b_2 \neq 0$ and $b_1 \neq 0$ are more complicated since

$$\Delta_1 F = \Delta_1 \delta \left\{ -k_1 b_2 + k_2 \left[b_2 (\gamma + 2D) - b_1 \right] \right\}$$

$$= \Delta_1 \delta \left\{ b_2 \left[(\gamma + 2D) k_2 - k_1 \right] - b_1 k_2 \right\}$$

However, one can choose k_1 and k_2 or, better, the ratio (k_1/k_2) so that $\Delta_1 F = 0$. This happens if the following relation holds:

$$\frac{k_1}{k_2} = (\gamma + 2D) - \frac{b_1}{b_2}$$

Whether this is a good choice still has to be investigated, particularly since one has to count on a controlled system whose coefficients b_1, b_2, D, and γ may change during operation.

It remains to be determined whether one is justified in using a switching function which has discontinuities. It is obvious that for systems with zeros, switching functions of the type $F = e + k_1 e' + k_2 e''$ generally will be discontinuous. However, by feeding δ back into the switching function, that is, forming a function F*

$$F^* = e + k_1 e' + k_2 e'' + f(\delta)$$

such discontinuities can be avoided. Whether this is an advantage for obtaining a fast follow-up with small errors still has to be investigated.

Stanford University,
Stanford, Calif., January 31, 1959.

APPENDIX A

MOTION AFTER AN END POINT IN PHASE SPACE

As has been indicated in figure 4(a), the system creeps toward the origin after an end point if a small time delay is present at the contactor. The average motion during this time is governed by the equation of the switching plane

$$F = e_{av} + k_1 e_{av}' + k_2 e_{av}'' = 0 \tag{A1}$$

End points occur only if $k_2 > 0$, because in this case the discontinuity in the slope of $dF/d\tau$ at $F = 0$ may not allow a continuation of the motion in an ideal system. Positive k_1 is advised for anticipating future changes of e. Therefore, only solutions of

$$e_{av} + k_1 e_{av}' + k_2 e_{av}'' = 0$$

for $k_1 > 0$ and $k_2 > 0$ have to be considered.

The equation (A1) may be written

$$e_{av}'' + \frac{k_1}{k_2} e_{av}' + \frac{1}{k_2} e_{av} = 0 \tag{A2}$$

and the solutions are

$$e = A_1 e^{\lambda_1 \tau} + A_2 e^{\lambda_2 \tau} \tag{A3}$$

with

$$\lambda_{1,2} = -\frac{k_1}{2k_2} \pm \frac{\sqrt{k_1^2 - 4k_2}}{2k_2} \tag{A4}$$

For

$$k_1^2 = 4k_2$$

there is a double root.

If $k_1^2 > 4k_2$, there are two real roots and for $k_1^2 < 4k_2$ there are two conjugate complex roots. In case of two complex roots (this is the the case which occurs, for instance, in the so-called "quasi-optimum" systems) a convenient plot of E_1 and E_2 using spirals could be done only if a new time unit were introduced:

$$\tau = \beta\tau^* \tag{A5}$$

Then equation (A2) leads to

$$\frac{d^2 e_{av}}{d\tau^{*2}} + \frac{k_1}{k_2}\beta\frac{de_{av}}{d\tau^*} + \frac{\beta^2}{k_2}e_{av} = 0$$

With $\beta^2 = k_2$, one obtains

$$\frac{d^2 e_{av}}{d\tau^{*2}} + \frac{k_1}{k_2}\sqrt{k_2}\frac{de_{av}}{d\tau^*} + e_{av} = 0$$

or

$$\frac{d^2 e_{av}}{d\tau^{*2}} + 2D^*\frac{de_{av}}{d\tau^*} + e_{av} = 0 \tag{A6}$$

where

$$D^* = \frac{k_1}{2\sqrt{k_2}}$$

In phase planes whose axes form the angle $(\pi - \sigma^*)$ with $\cos\sigma^* = -D^*$ with each other, phase trajectories of the average motion are represented by logarithmic spirals.

The initial conditions for the spiral are given by the E_1 and E_2 values at the end point. After having designed the average motion in the new plane, the curve must be transferred into the original $E_1 - E_2$ plane with $\cos\sigma = -D$. This is best done point by point.

In the case of real roots, both roots will be negative. For real roots solution (A3) allows one to write down immediately a convenient relation between E_1 and E_2. With

$$E_{1_{av}} = \gamma e_{av}' + e_{av}''$$

and

$$E_{2_{av}} = \gamma e_{av} + e_{av}'$$

one obtains

$$E_{1_{av}} = C_1\lambda_1 e^{\lambda_1\tau} + C_2\lambda_2 e^{\lambda_2\tau}$$

and

$$E_{2_{av}} = C_1 e^{\lambda_1\tau} + C_2 e^{\lambda_2\tau}$$

By eliminating τ from these two expressions, the equation for the $E_1 - E_2$ trajectory becomes

$$(E_1 - \lambda_1 E_2)^{\lambda_1} = M^*(E_1 - \lambda_2 E_2)^{\lambda_2} \tag{A7}$$

The following example shows the construction of the after-end-point motion in the phase space:

Let the system be determined by $N = 25$, $D = 0.2$, $\gamma = 0$, $k_1 = 1.0$, and $k_2 = 0.5$.

The initial conditions are given as

$$E_{10} = 0$$

$$E_{20} = 20$$

$$E_{30} = 0$$

or

$$e_0 = -8$$

$$e_0' = 20$$

$$e_0'' = 0$$

The system reaches an end point at the point

$$E_1 = -13.4$$

$$E_2 = 7.2$$

$$E_3 = -10.8$$

or

$$e = -0.3$$

$$e' = 7.2$$

$$e'' = -13.4$$

The average motion during chatter is given by the differential equation

$$e_{av}'' + 2e_{av}' + 2e_{av} = 0 \qquad \text{(A8)}$$

with the characteristic roots

$$\lambda_{1,2} = -1 \pm i$$

That means that the average motion is under-damped and that the use of logarithmic spirals simplifies the plotting. Therefore, equation (A8) is normalized by introducing

$$\tau = \beta\tau^*$$

with

$$\beta = \sqrt{k_2}$$

The damping coefficient turns out to be $D^* = 0.707$.

The trajectory of the average motion is first drawn in a normalized phase plane by using a logarithmic spiral and then denormalized point by point and plotted in the original phase space. Figure 17 shows the trajectory in the phase space. As is seen from the figure, the system is rapidly brought down to the vicinity of the origin.

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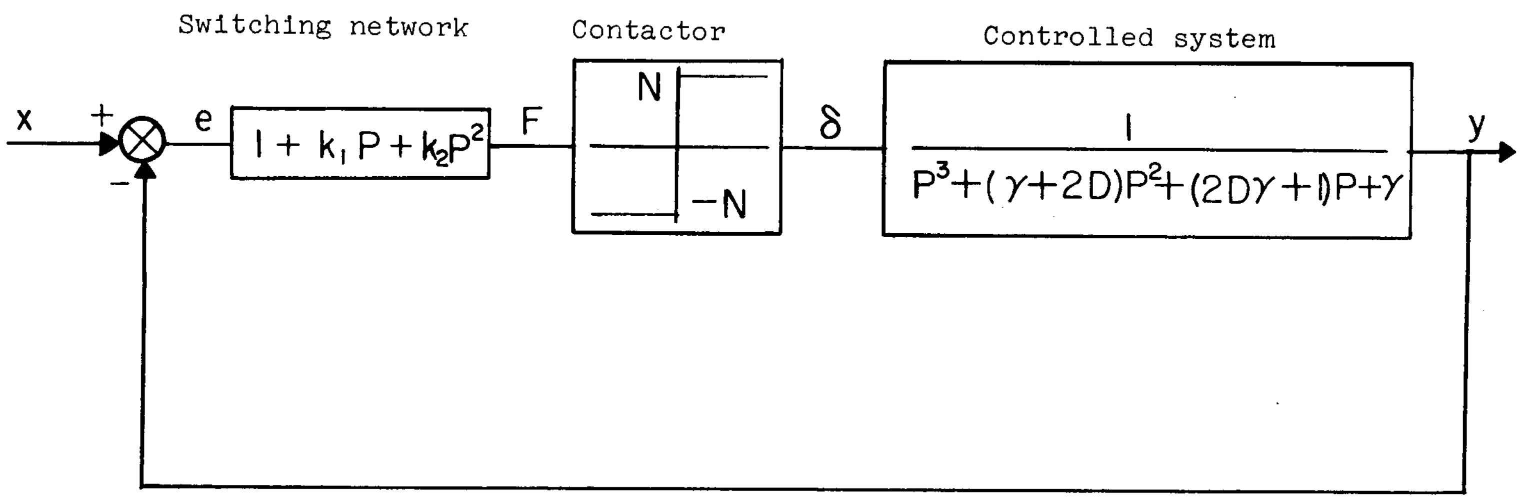


Figure 1.- Block diagram of the third-order contactor control system.

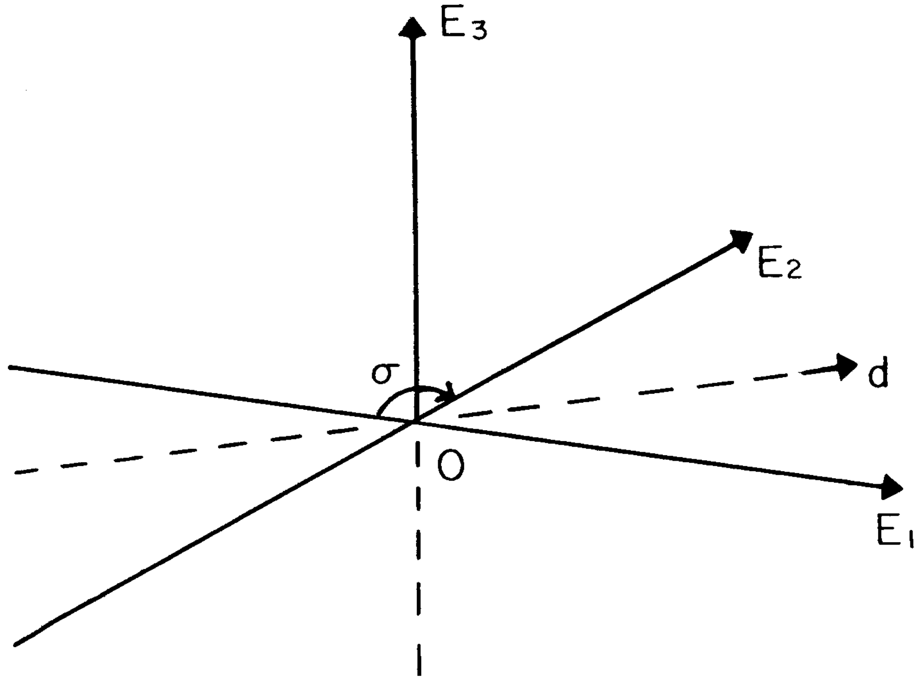
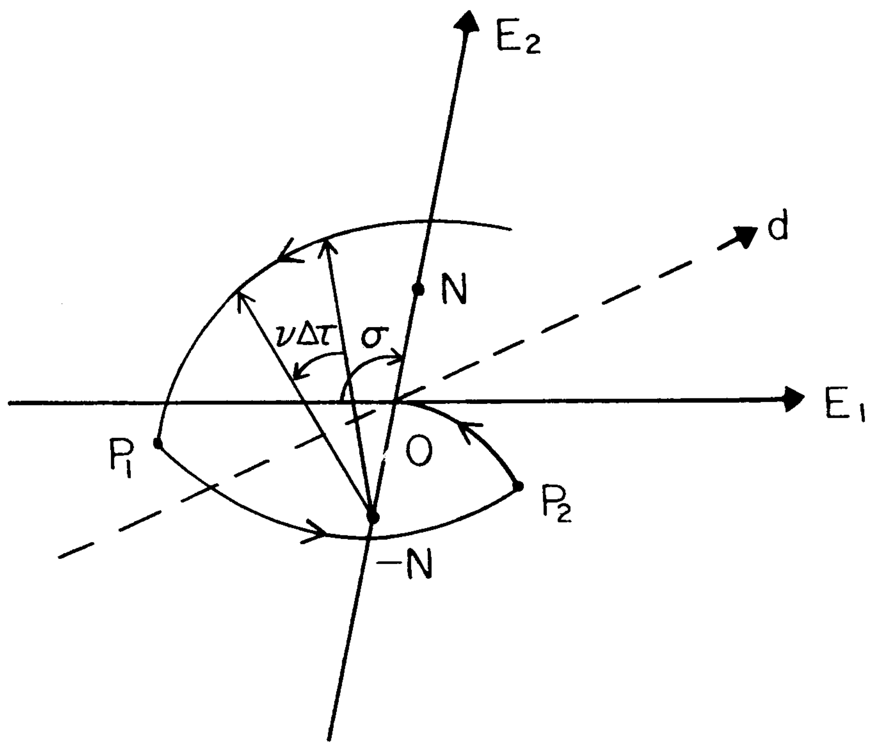
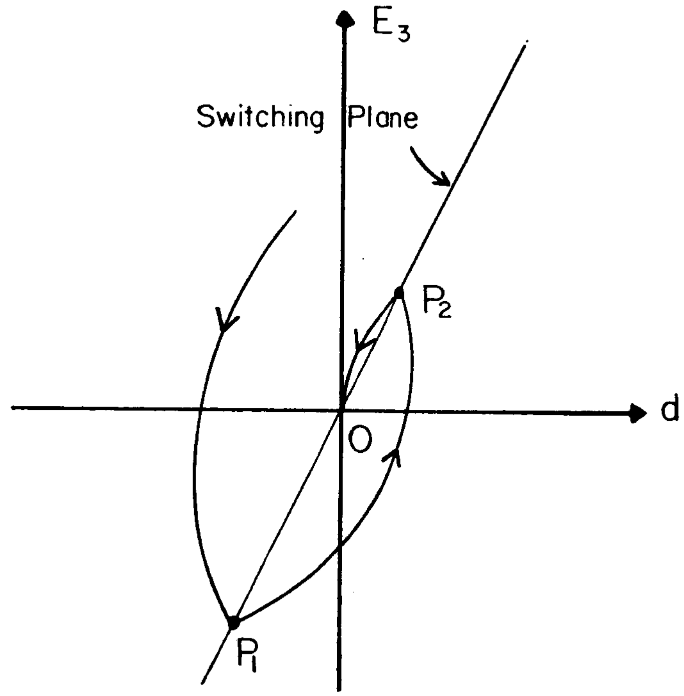


Figure 2.- Illustration of phase space.

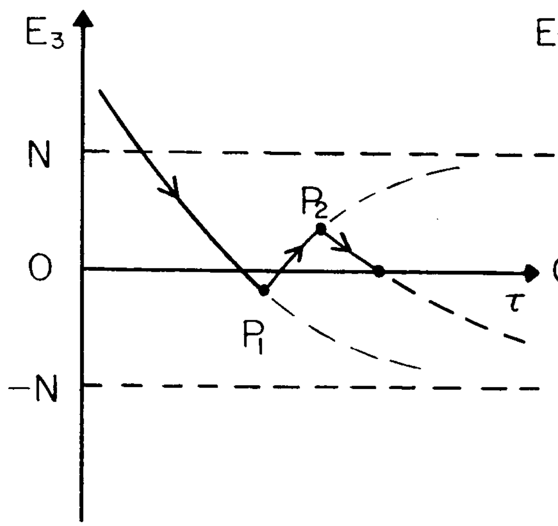


(a) Projection into E_1 - E_2 plane.

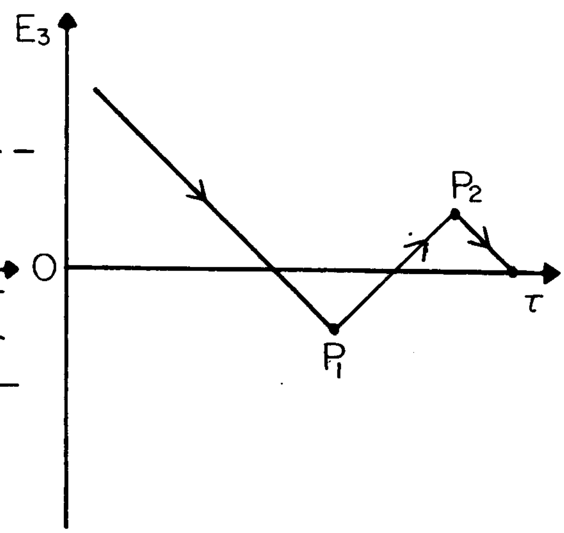
Figure 3.- Projection of trajectory.



(b) Projection into E_3d plane.

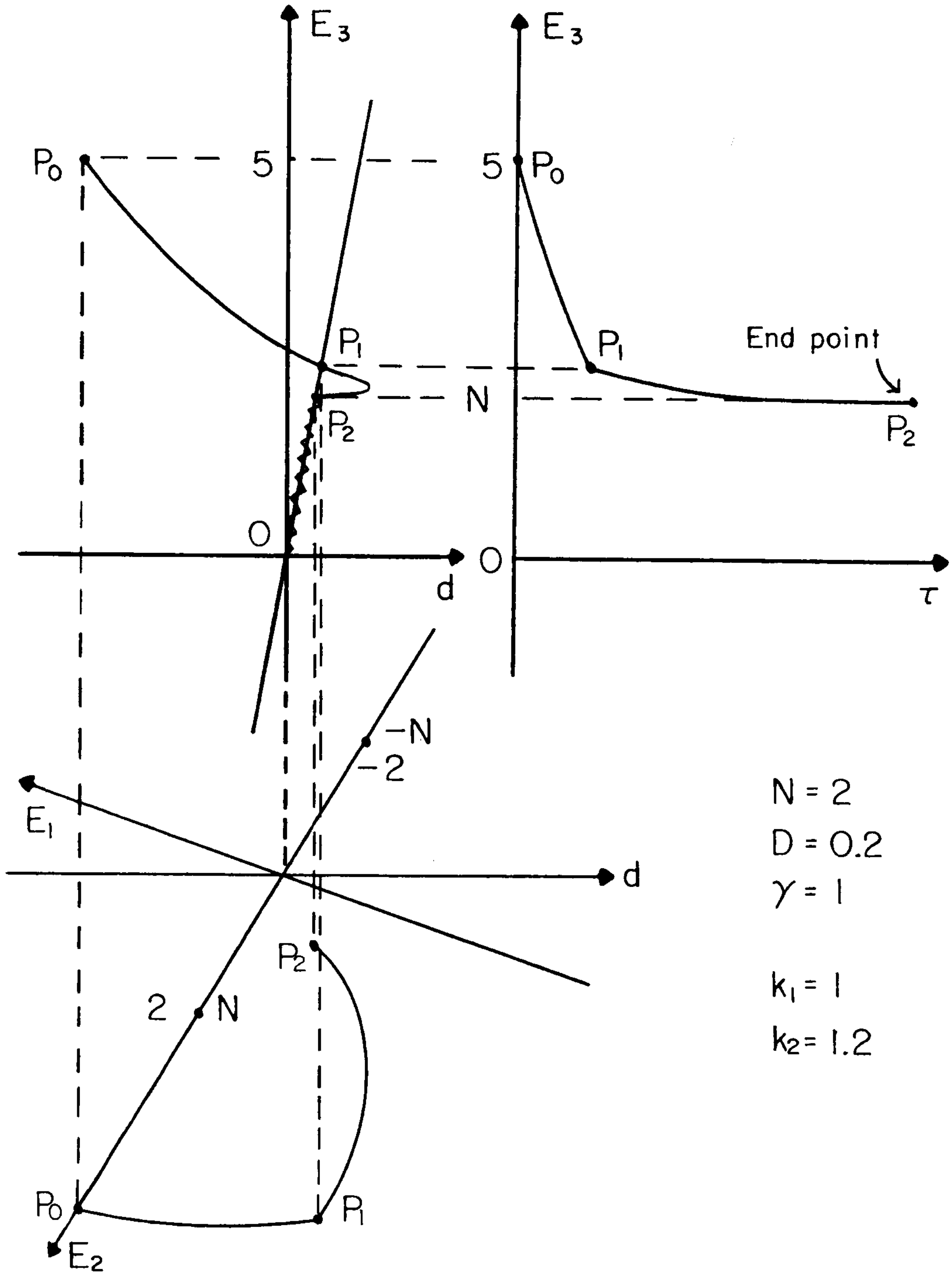


(c) Diagram of E_3 against τ; $\gamma \neq 0$.



(d) Diagram of E_3 against τ; $\gamma = 0$.

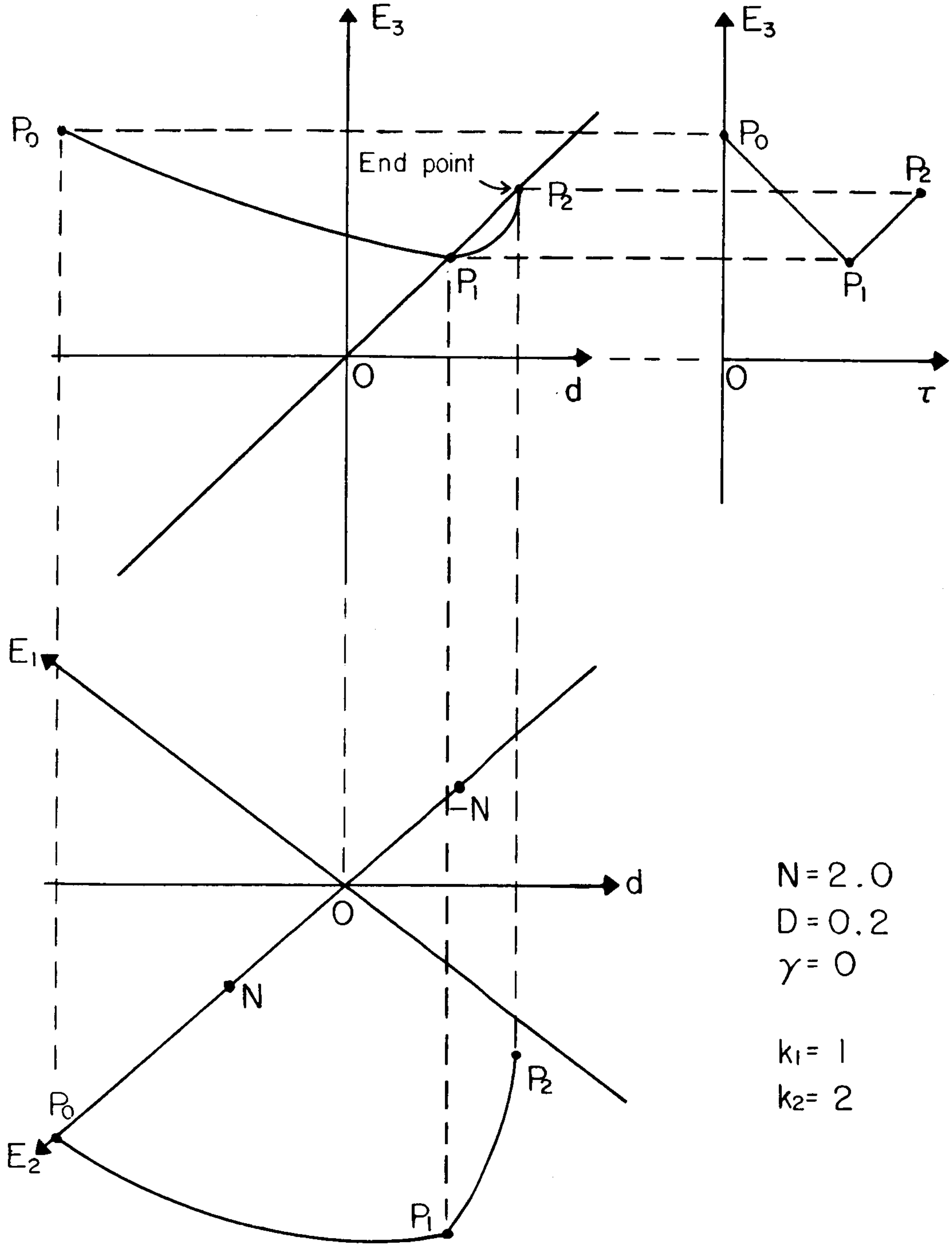
Figure 3.- Concluded.



(a) Example (1); $\gamma \neq 0$.

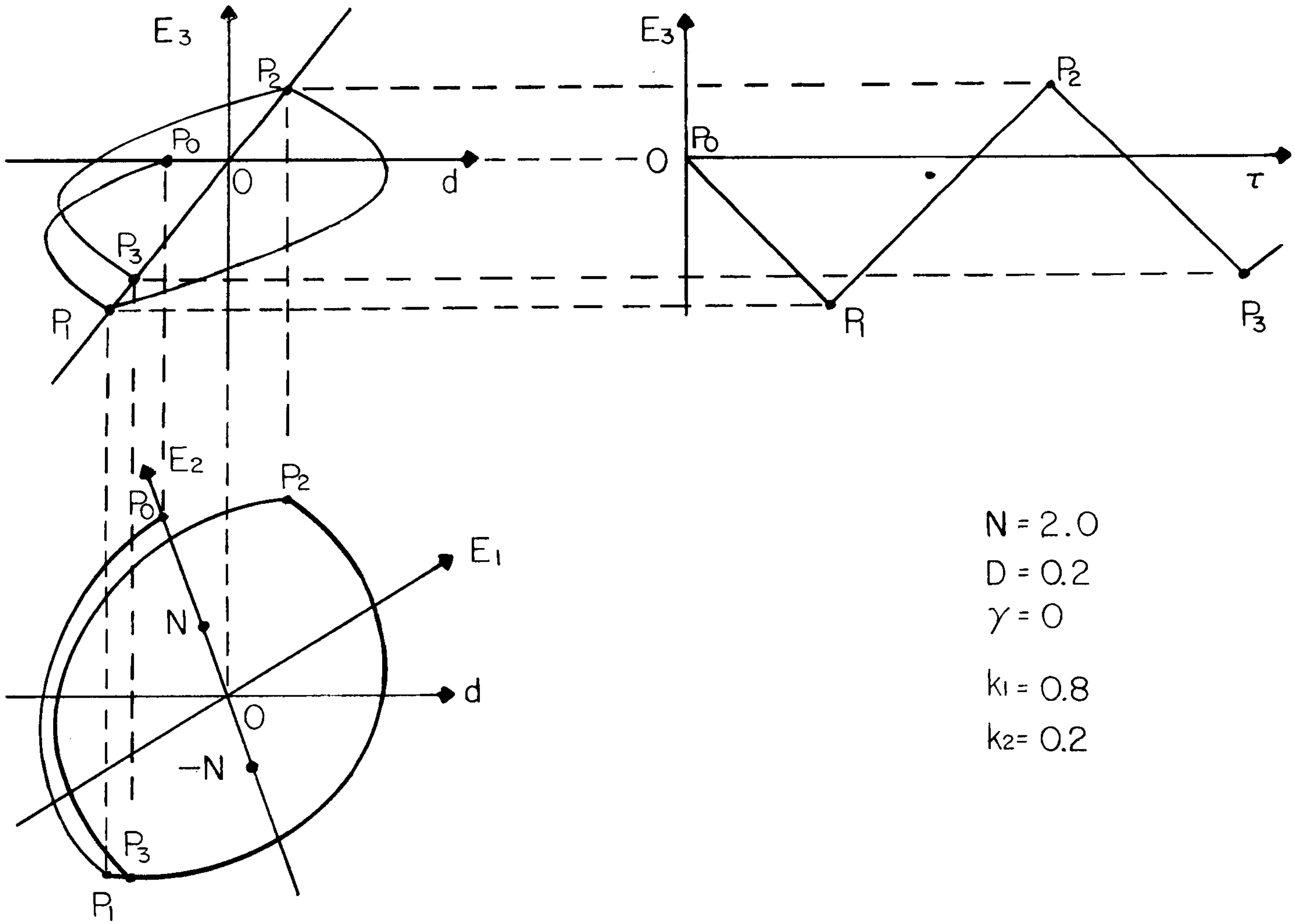
Figure 4.- Examples of motion in phase space.

W-138



(b) Example (2); $\gamma = 0$.

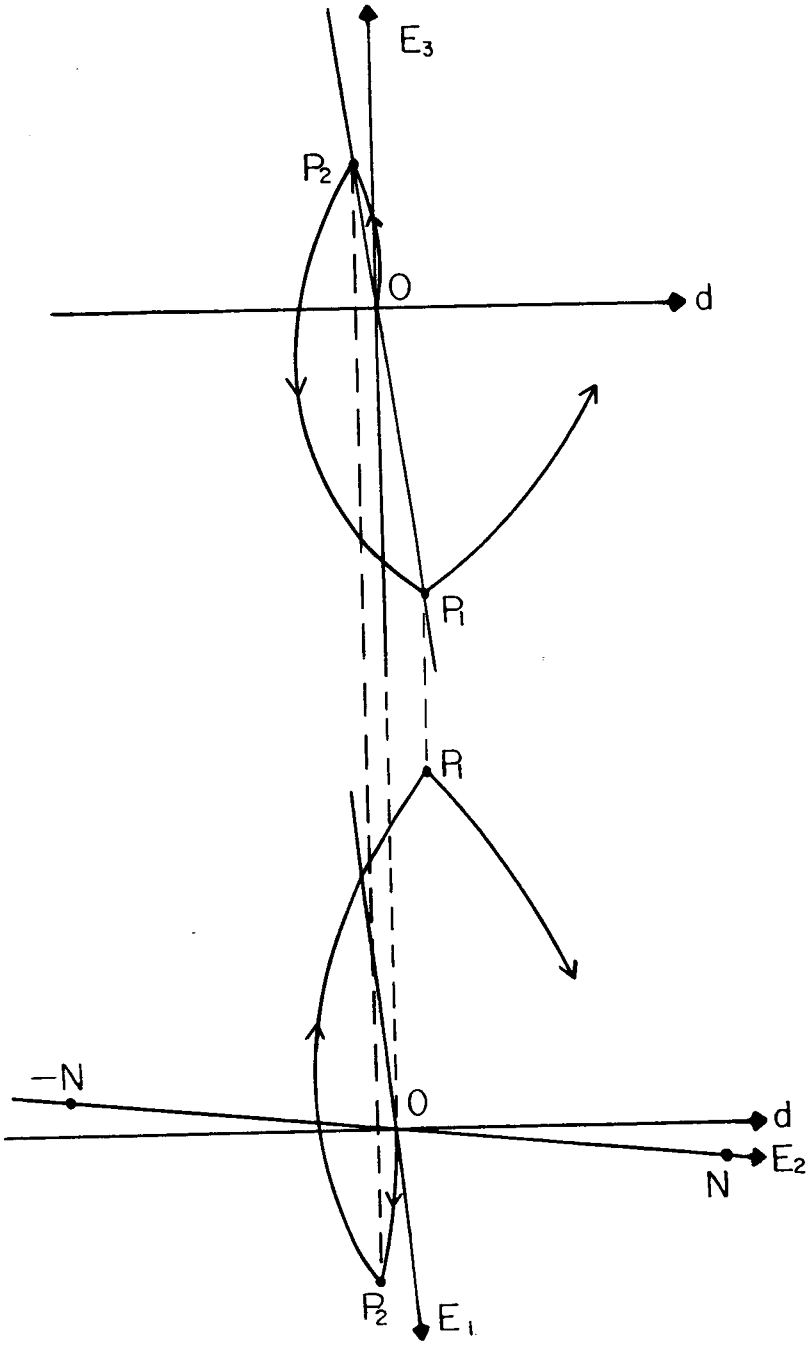
Figure 4.- Continued.



(c) Example (3); improper selection of switching plane; $\gamma = 0$.

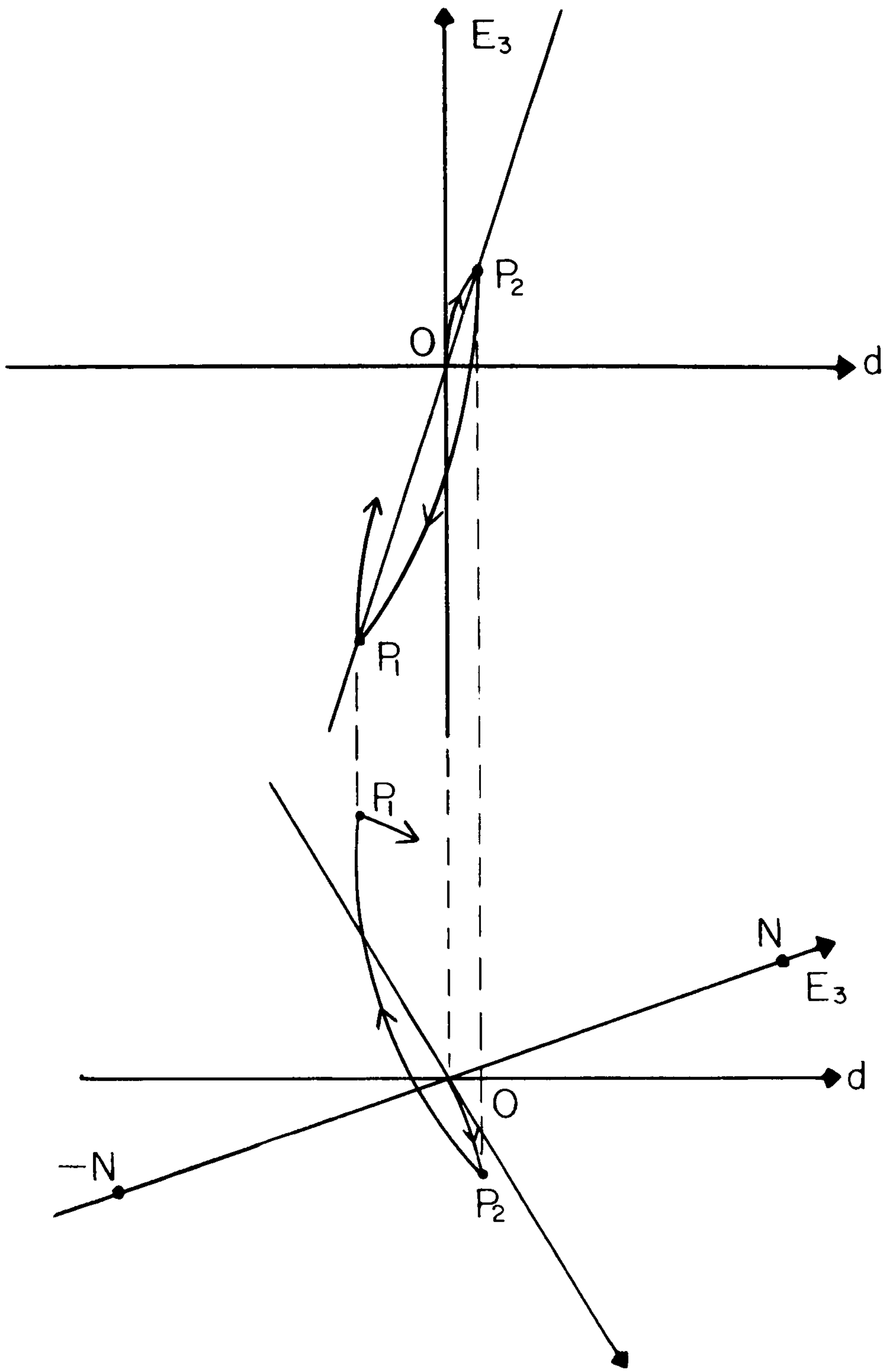
Figure 4.- Concluded.

W-138



(a) $\gamma = 0$.

Figure 5.- Example of optimum response. Trajectory was plotted backward; N = 2.0; D = 0.2.



(b) $\gamma = 1$.

Figure 5.- Concluded.

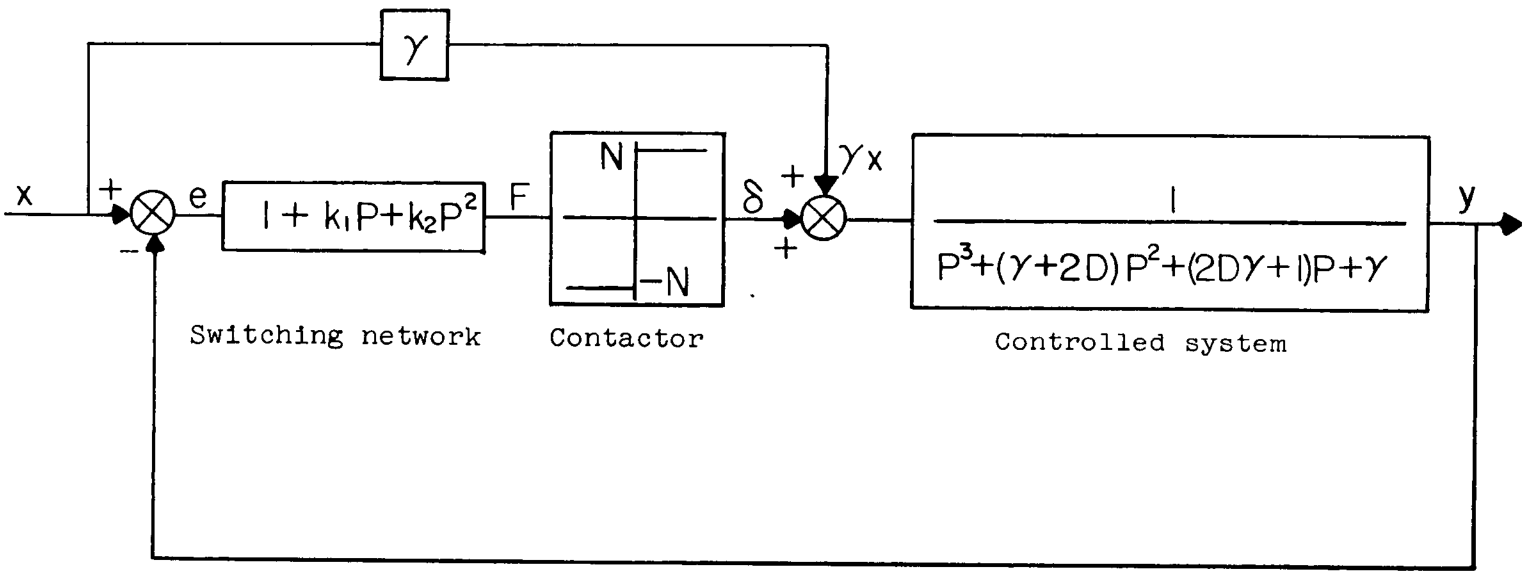
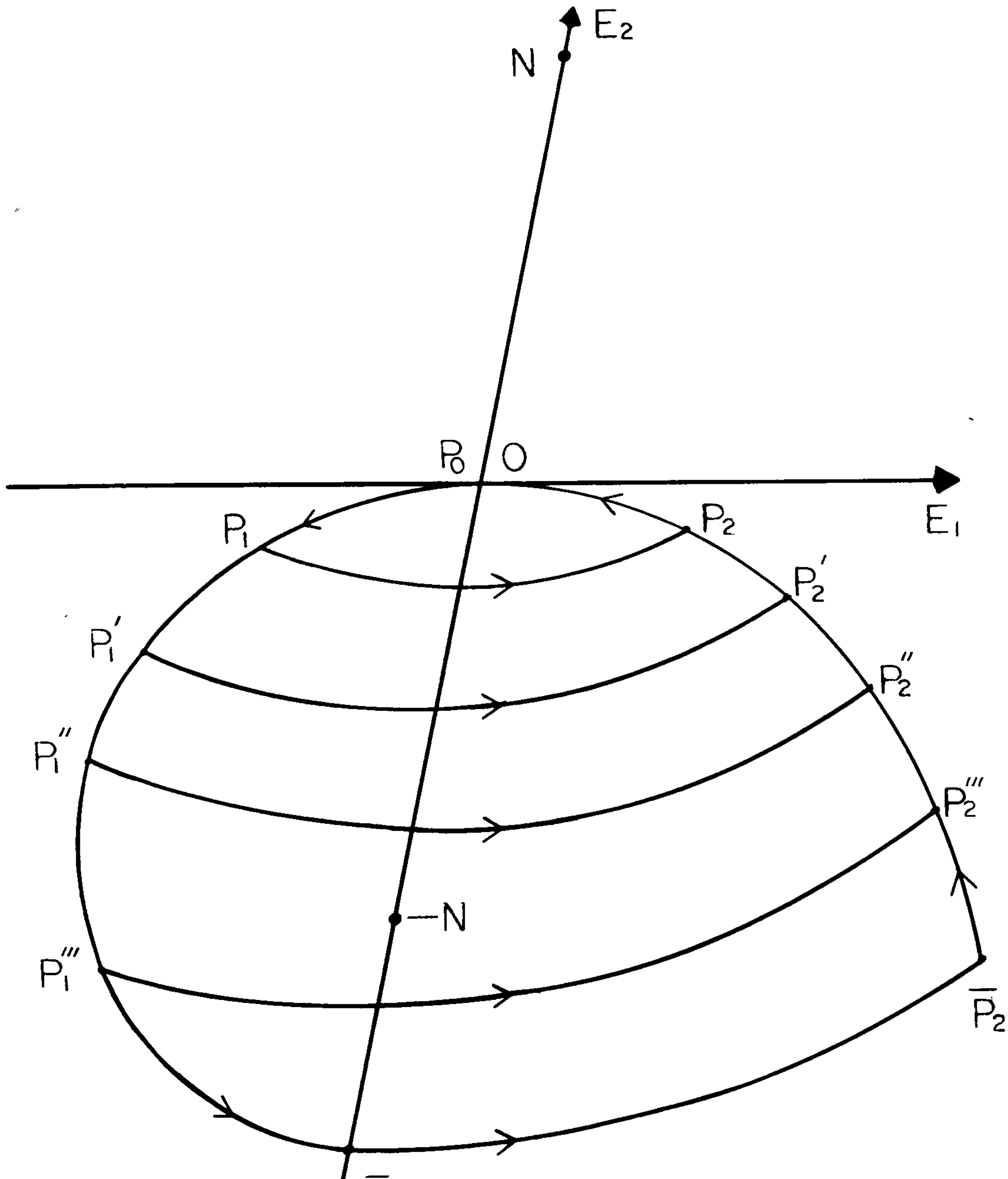
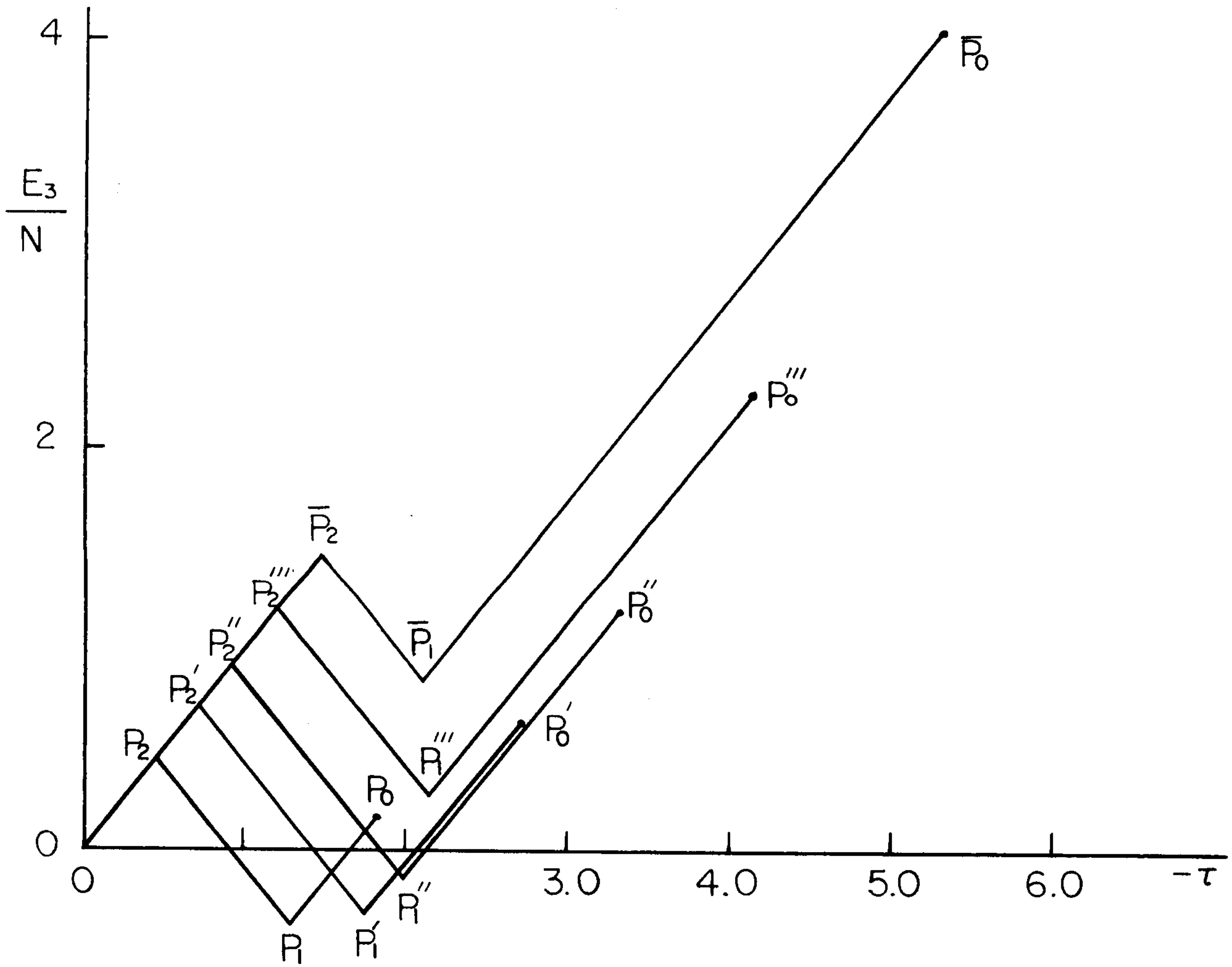


Figure 6.- Block diagram of third-order system with forward feed of input.



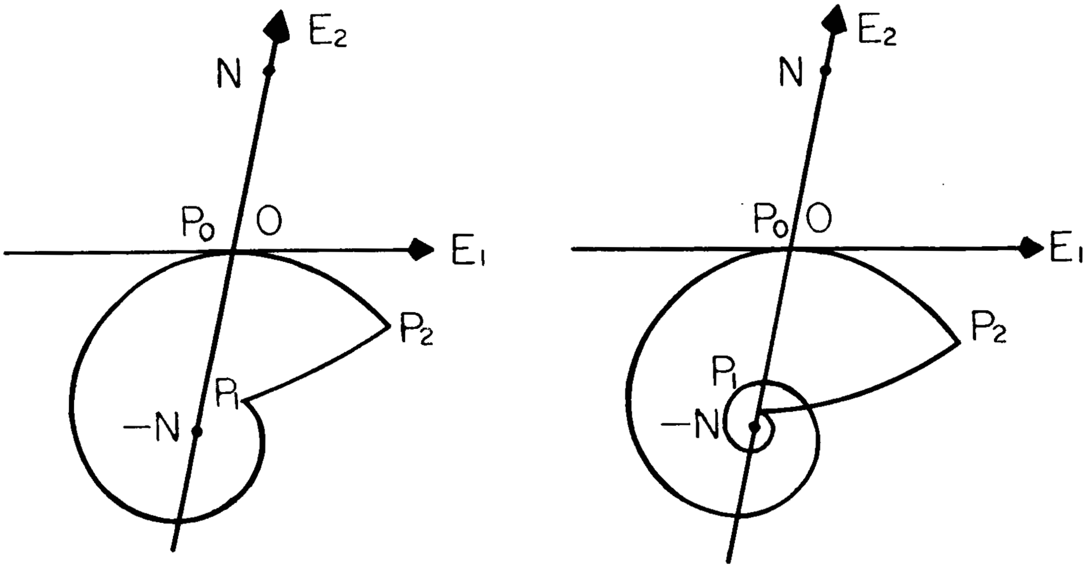
(a) Projections in E_1E_2 plane. $\gamma = 0$; D = 0.2; N = 25.

Figure 7.- Optimum responses to step inputs of different heights.

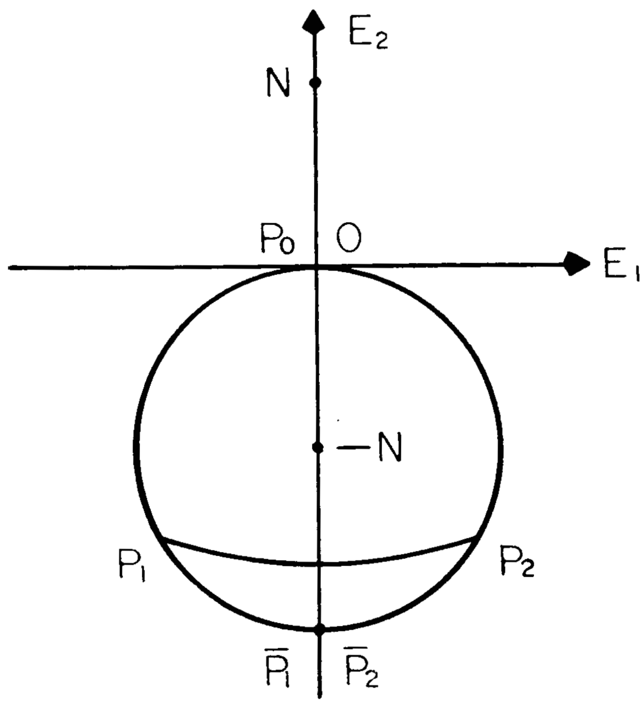


(b) Variation of E_3/N with time τ. $\gamma = 0$; $D = 0.2$.

Figure 7.- Continued.



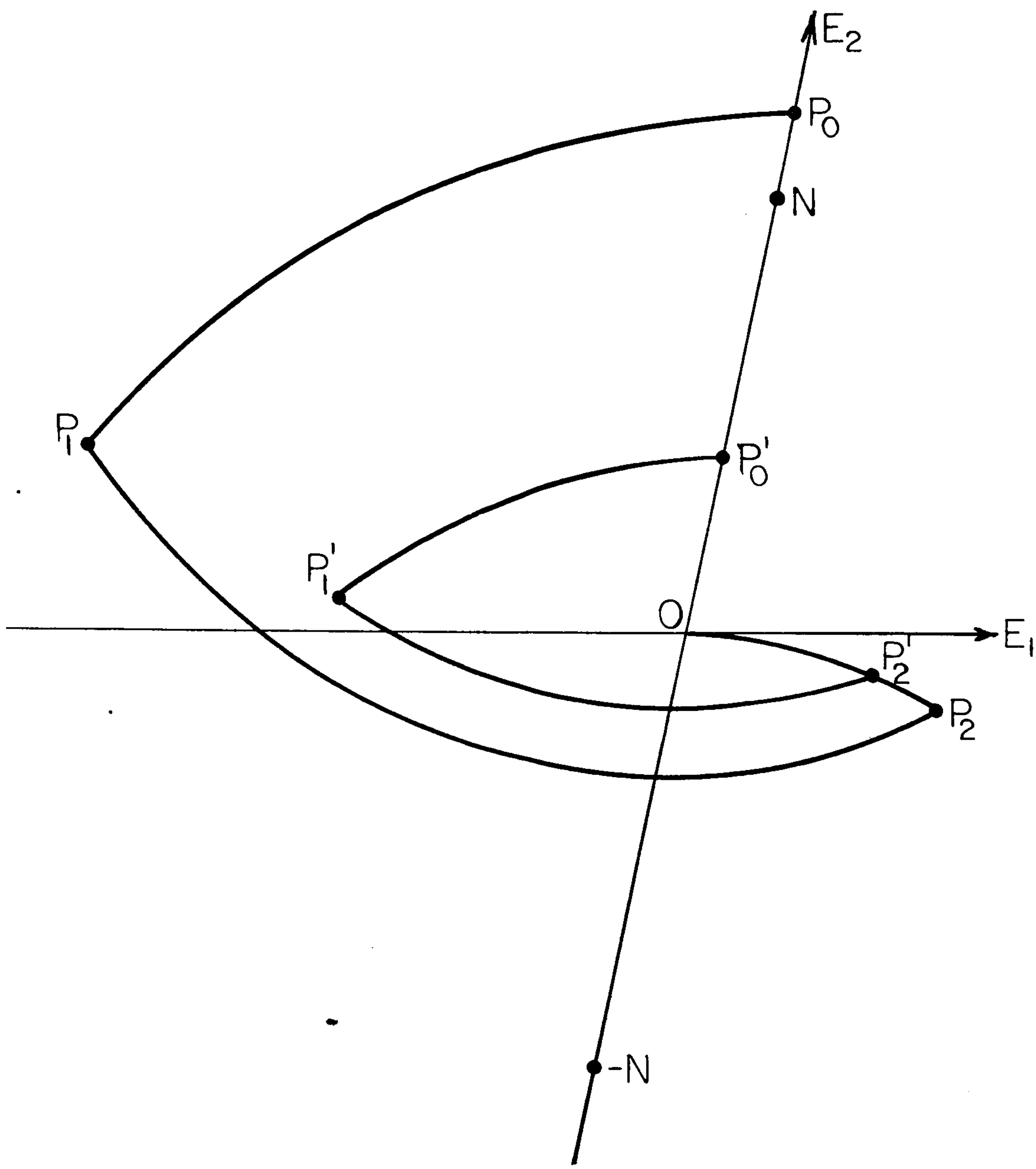
(c) Response to higher step inputs. $\gamma = 0$; $D = 0.2$.



(d) Response where $D = 0$ and $\gamma = 0$.

Figure 7.- Continued.

W-138



(e) Response to step input with $\gamma \neq 0$. $\gamma = 1$; D = 0.2; N = 25.

Figure 7.- Concluded.

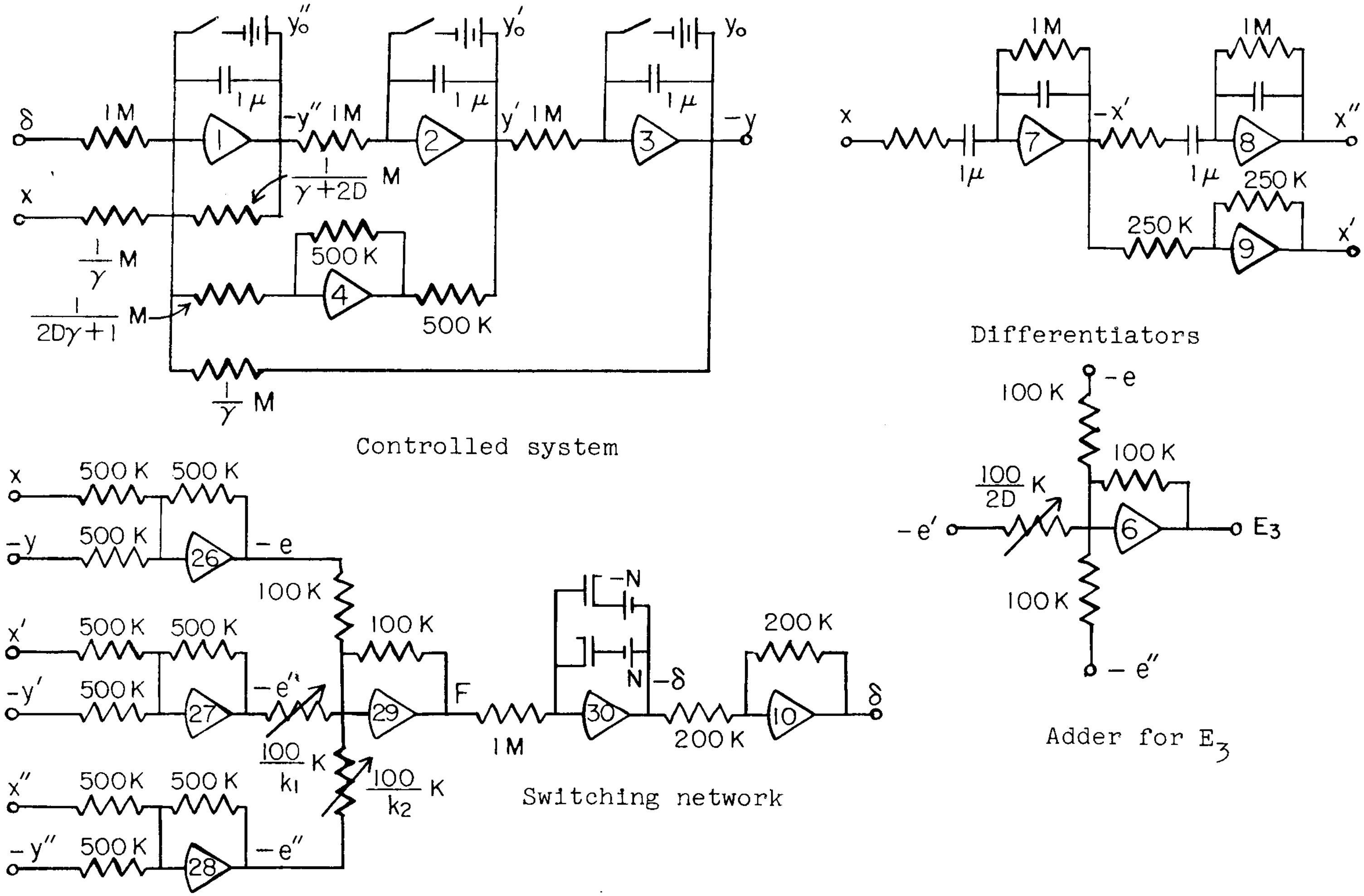
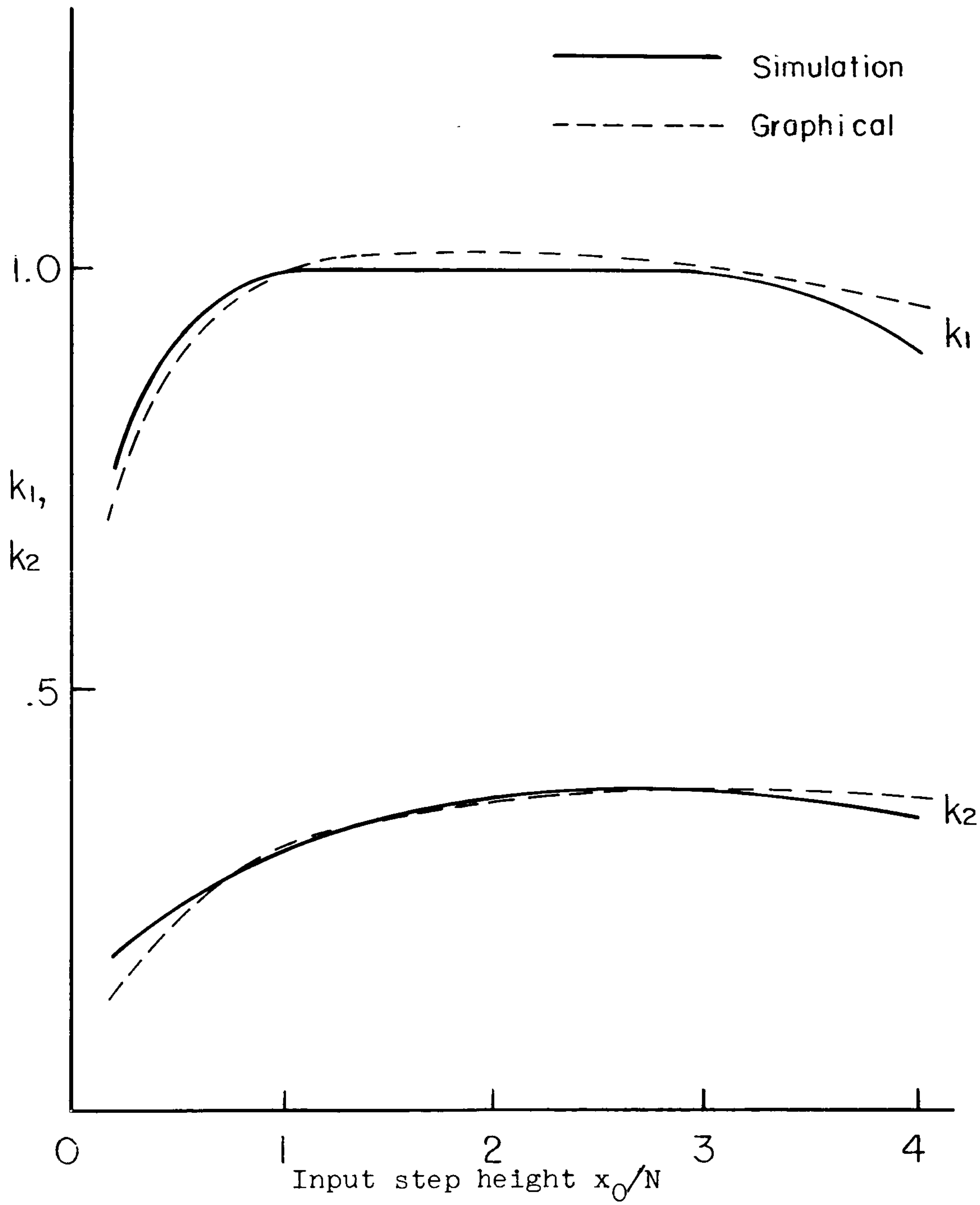
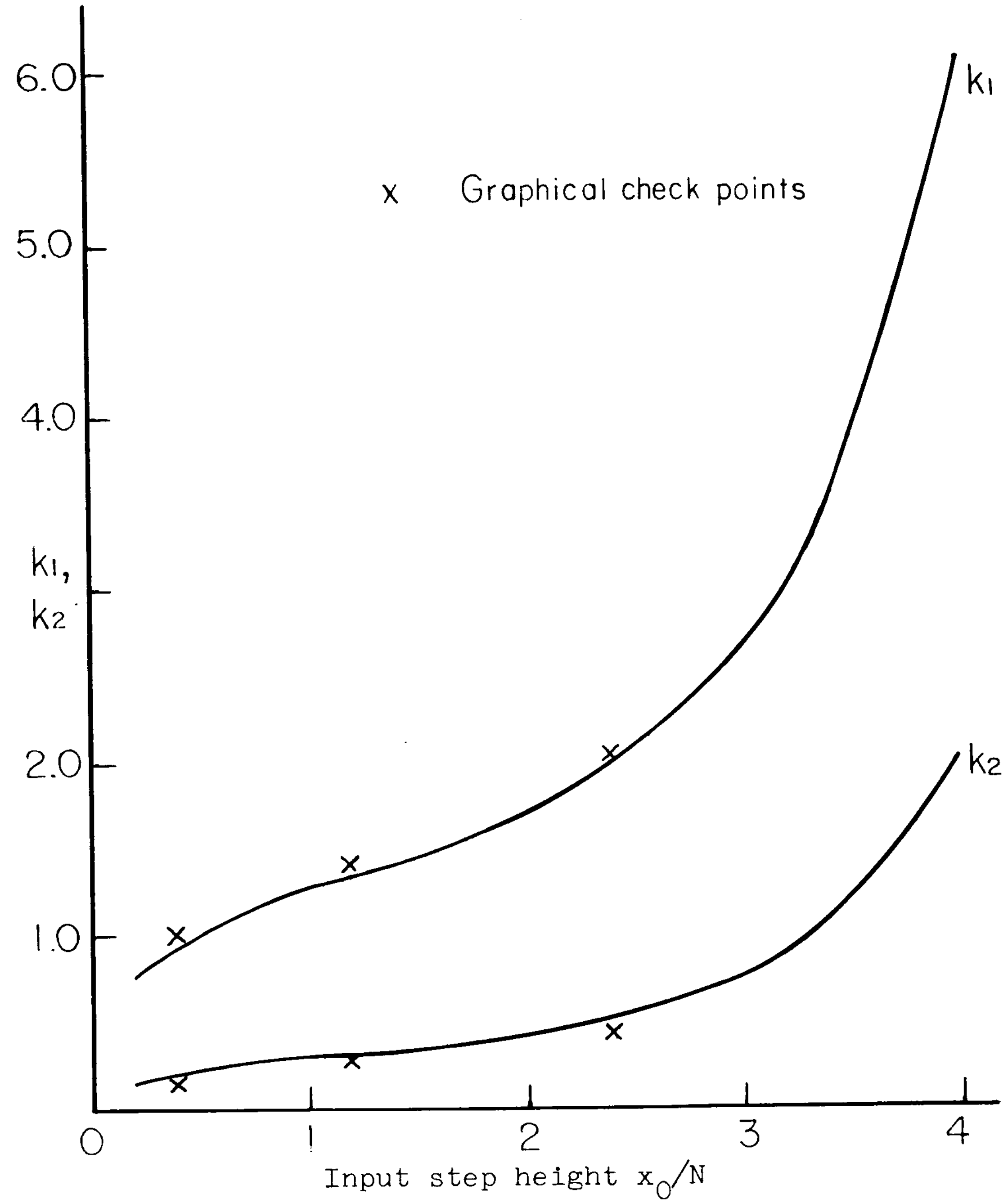


Figure 8.- Computer setup for third-order system.



(a) γ = 0; D = 0.2.

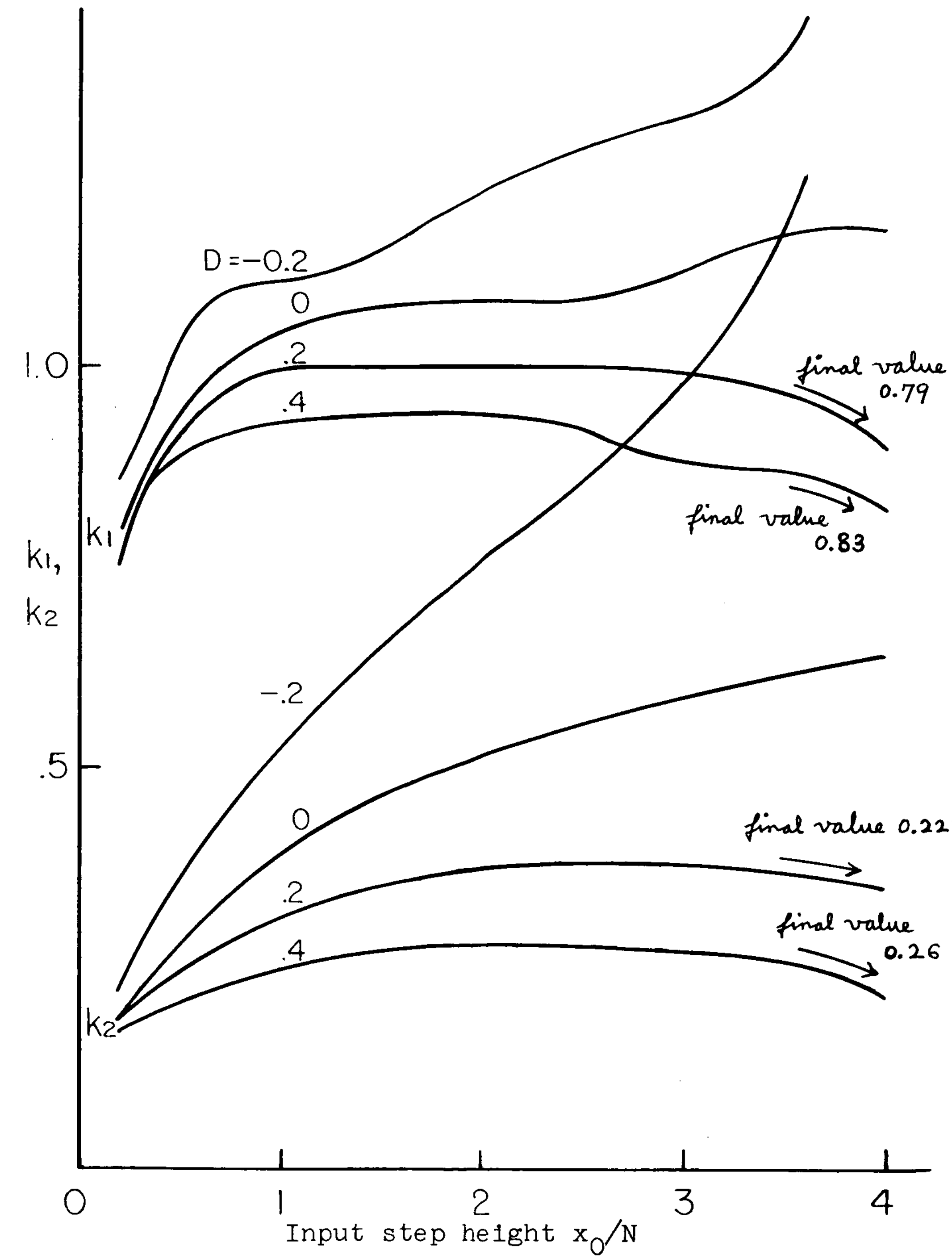
Figure 9.- Optimum values of k_1 and k_2 for various step inputs. N = 25.



(b) $\gamma = 1$; D = 0.2.

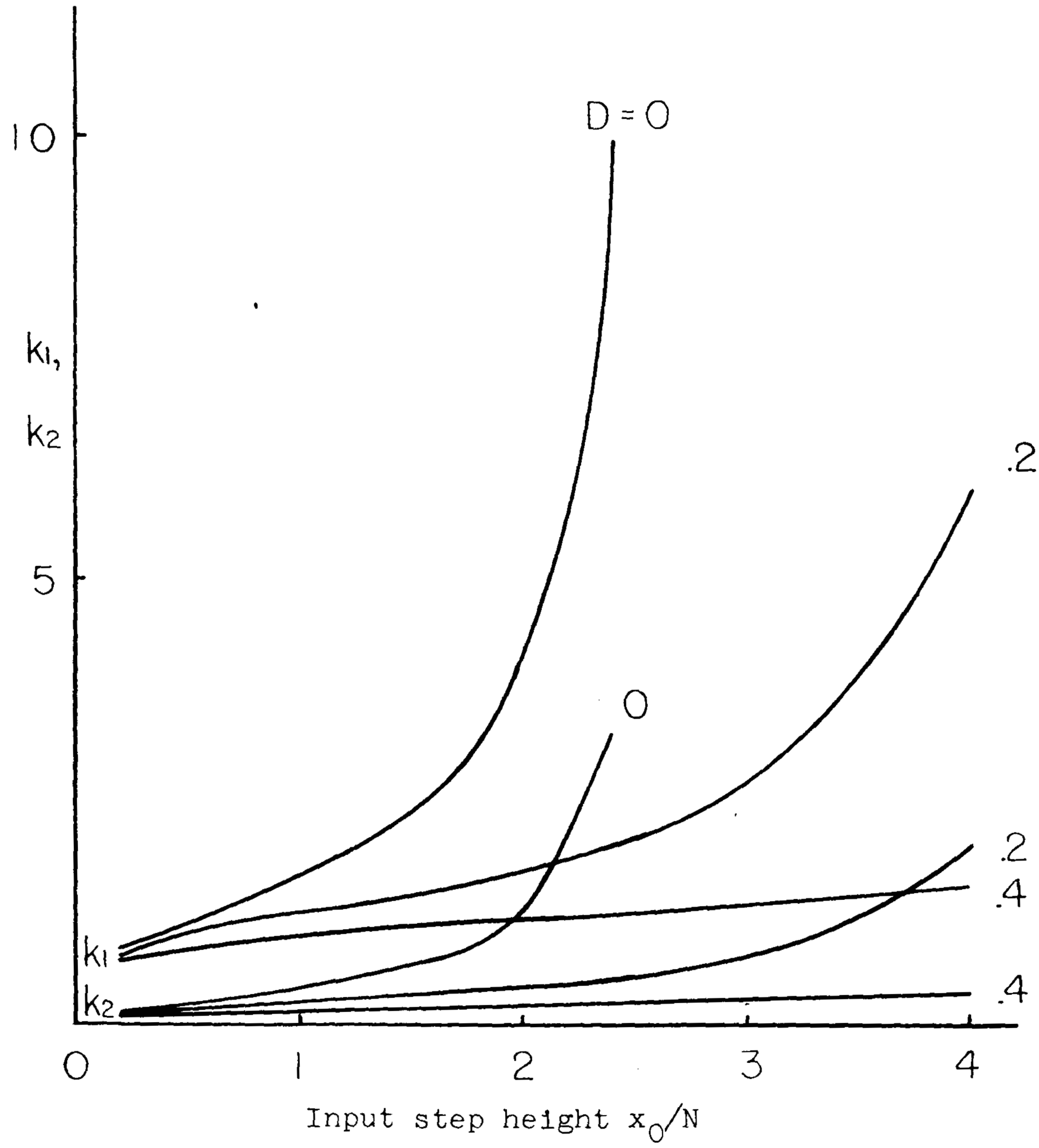
Figure 9.- Continued.

W-138



(c) $\gamma = 0$; various values of D.

Figure 9.- Continued.

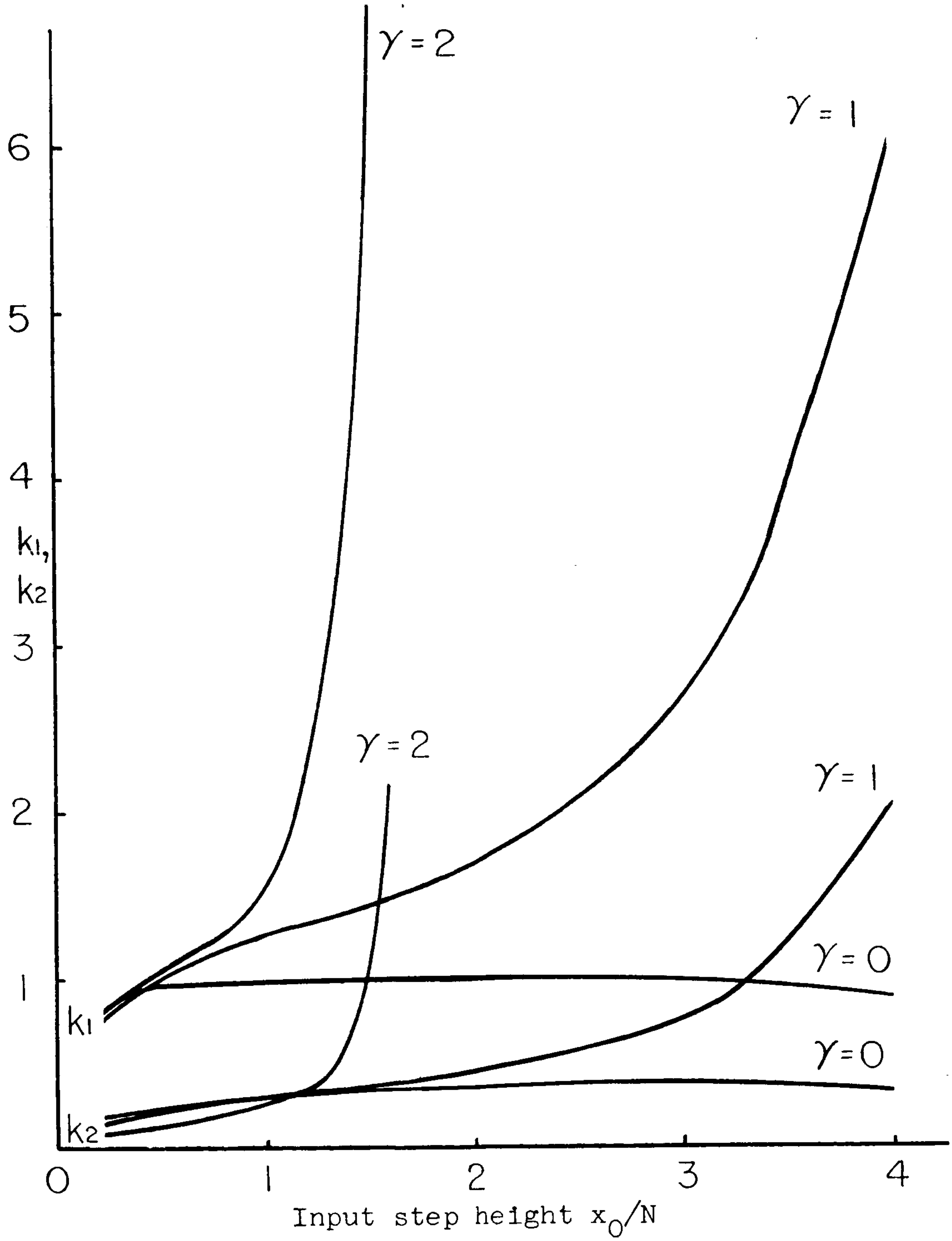


(d) $\gamma = 1$; various values of D.

Figure 9.- Continued.

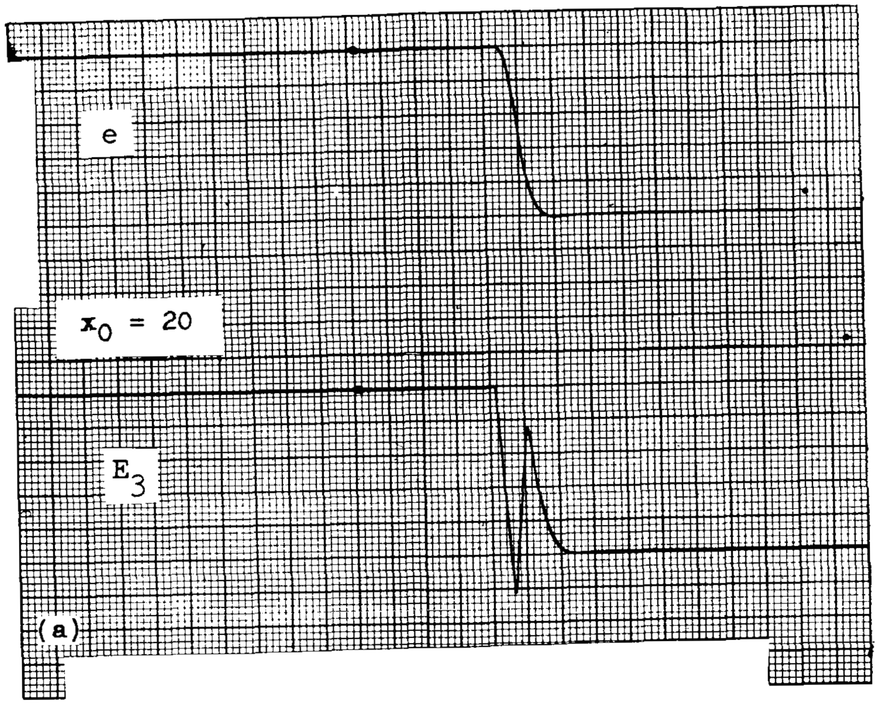
W-138

W-138



(e) Various values of γ; D = 0.2.

Figure 9.- Concluded.



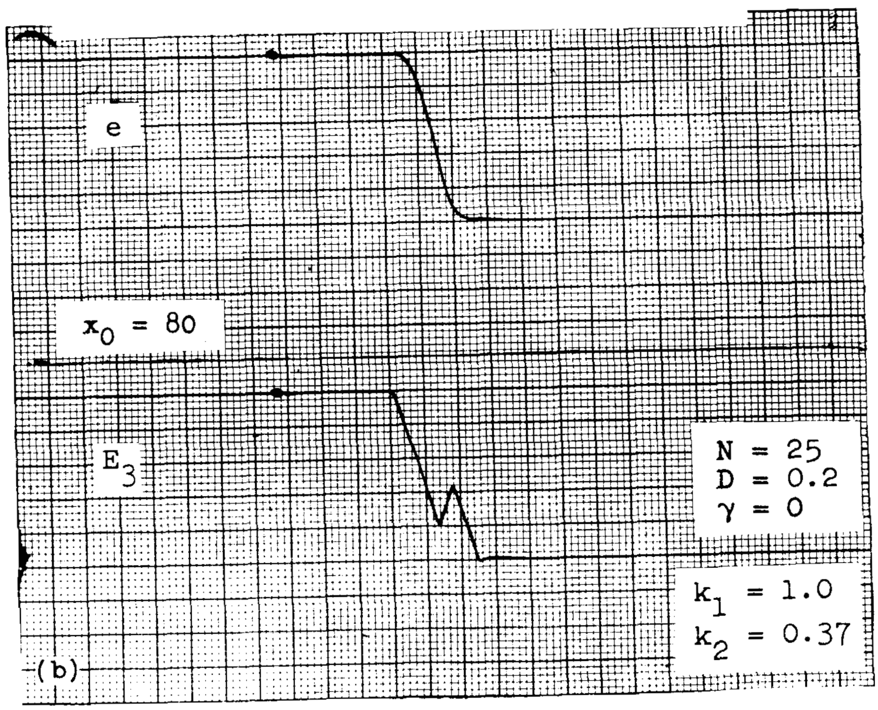


Figure 10.- Quasi-optimum response of third-order system.

W-138

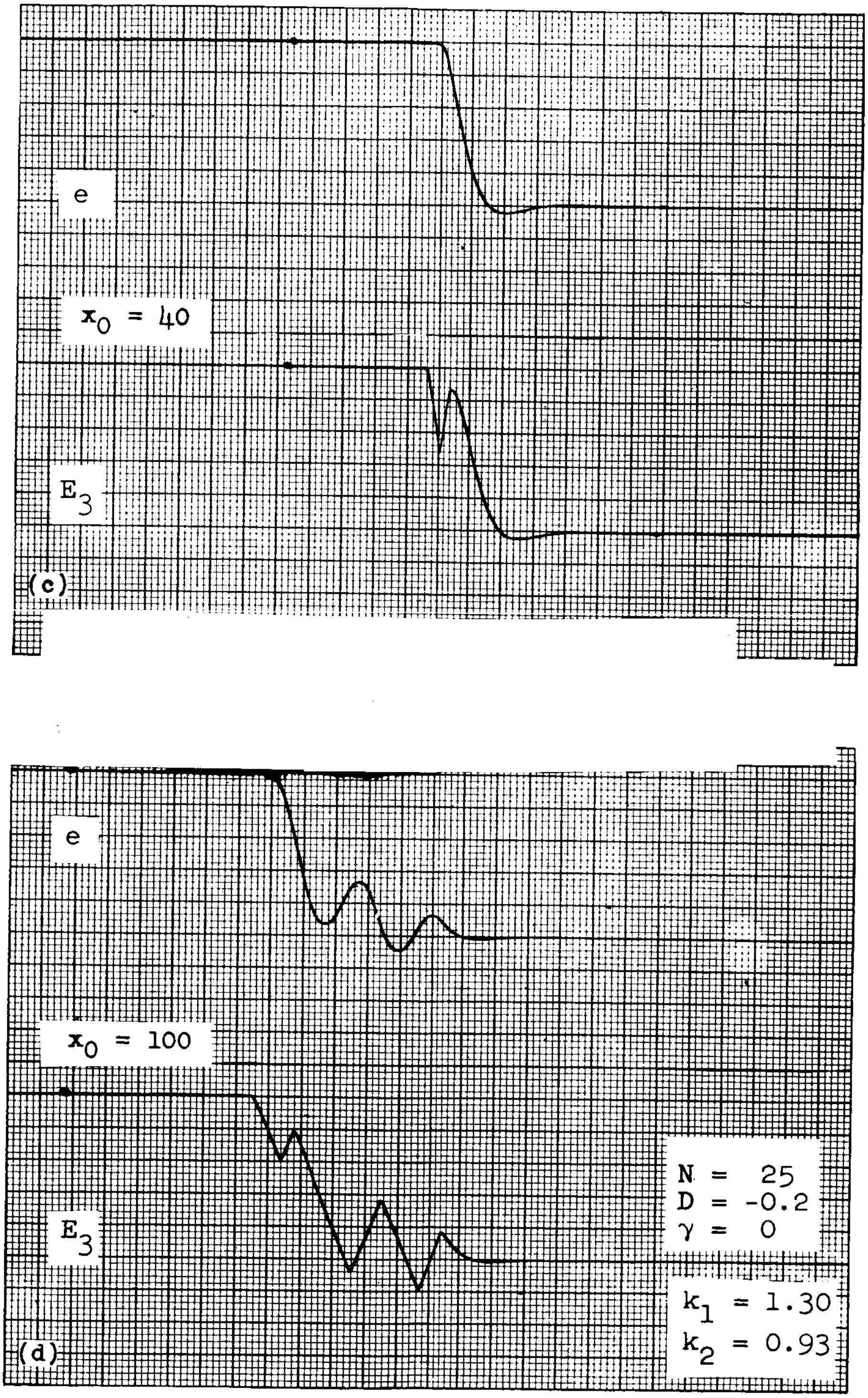
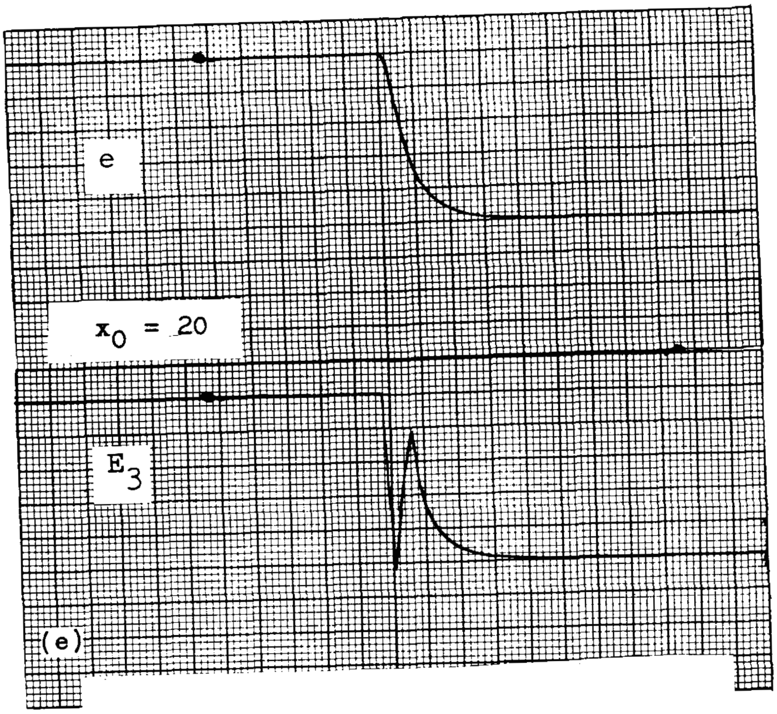


Figure 10.- Continued.



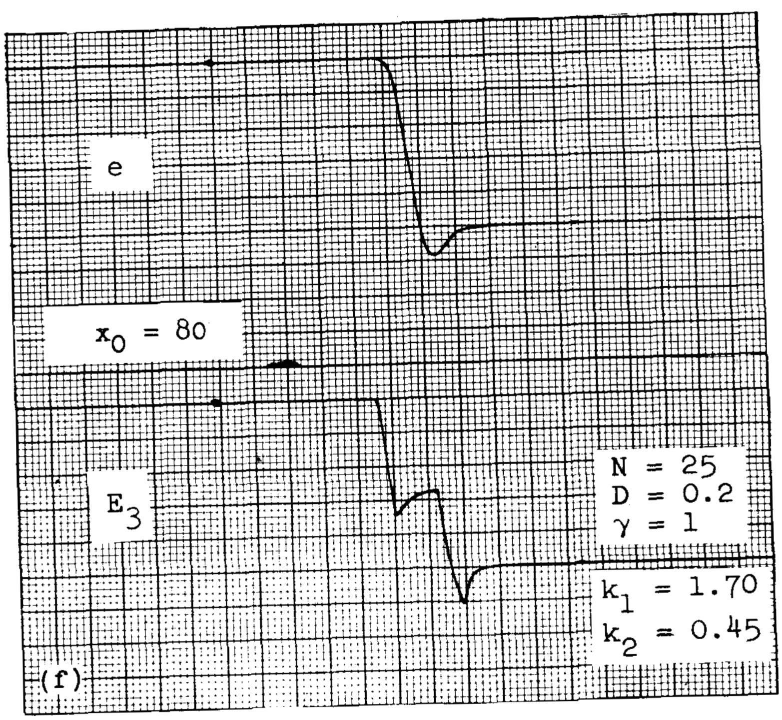
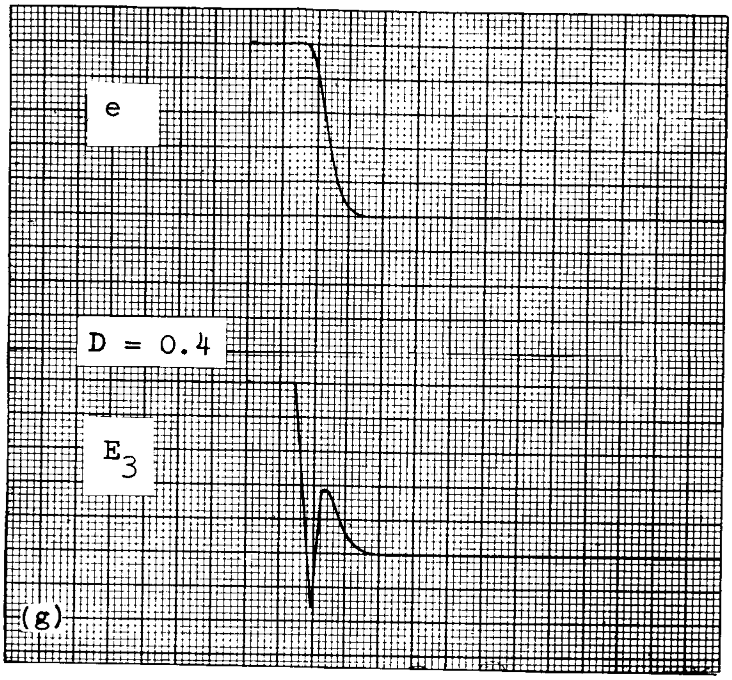


Figure 10.- Continued.

W-138



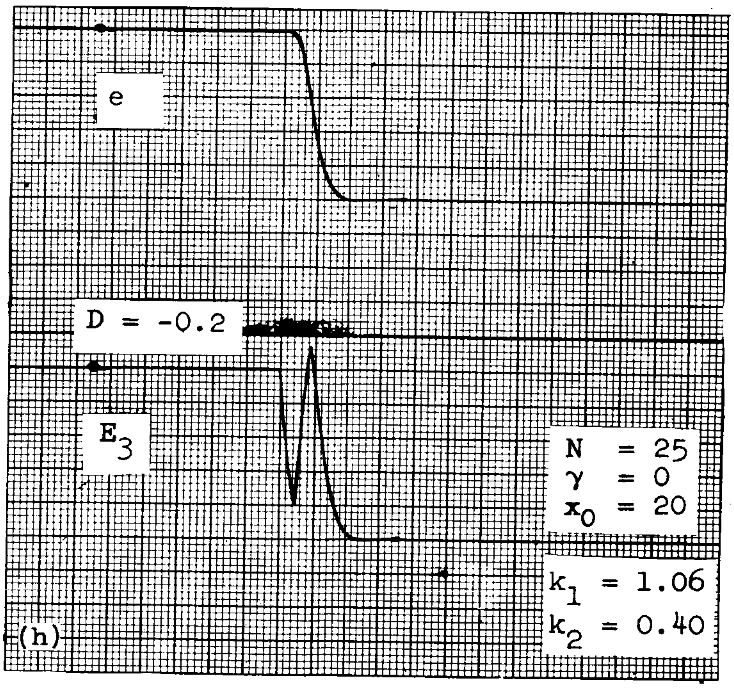
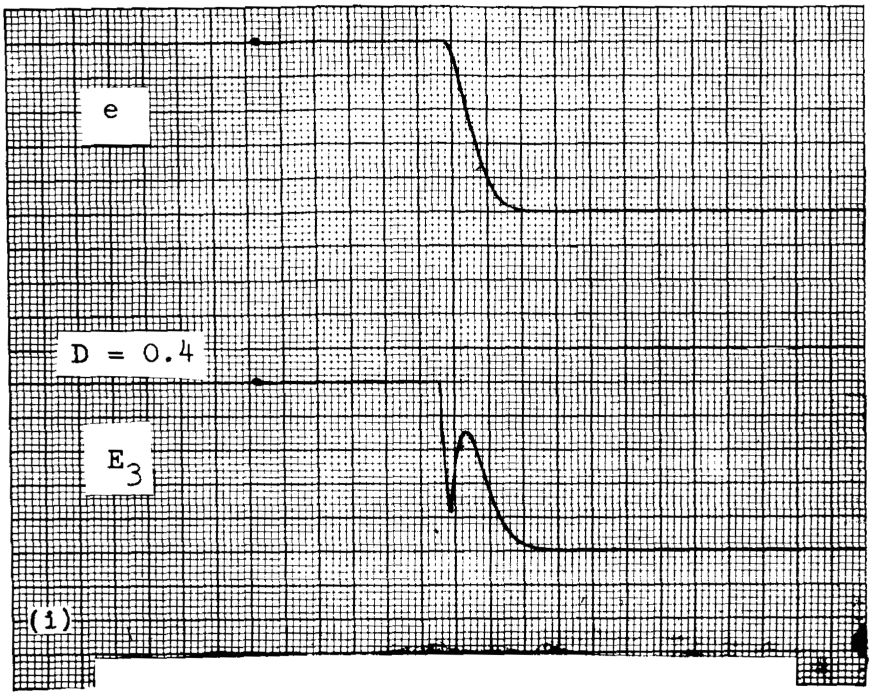


Figure 10.- Continued.



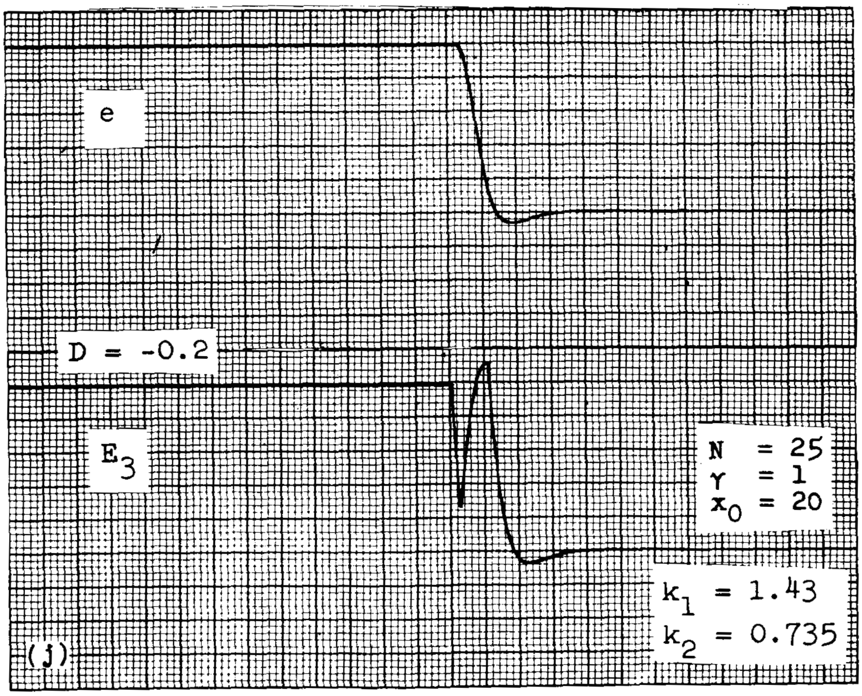
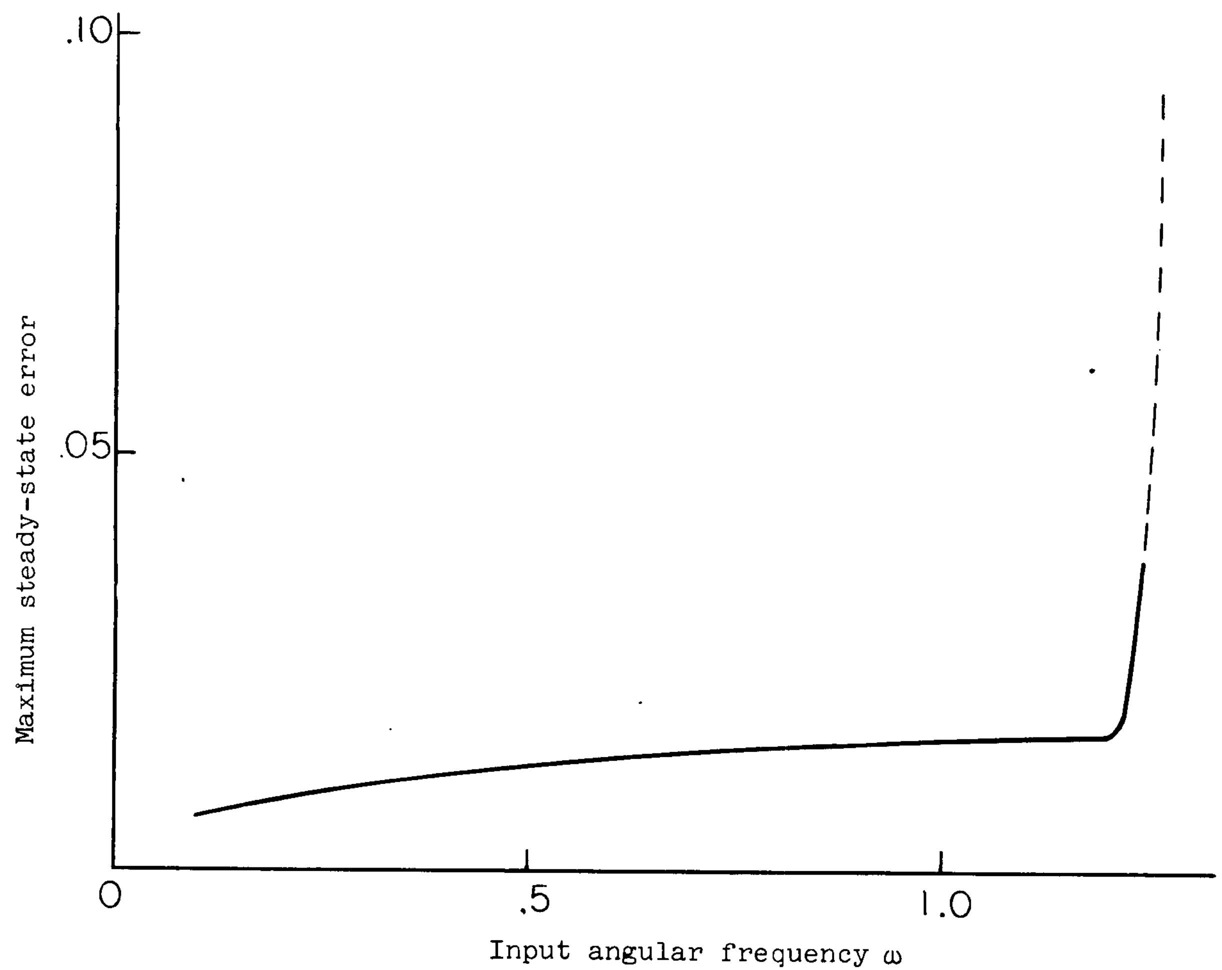
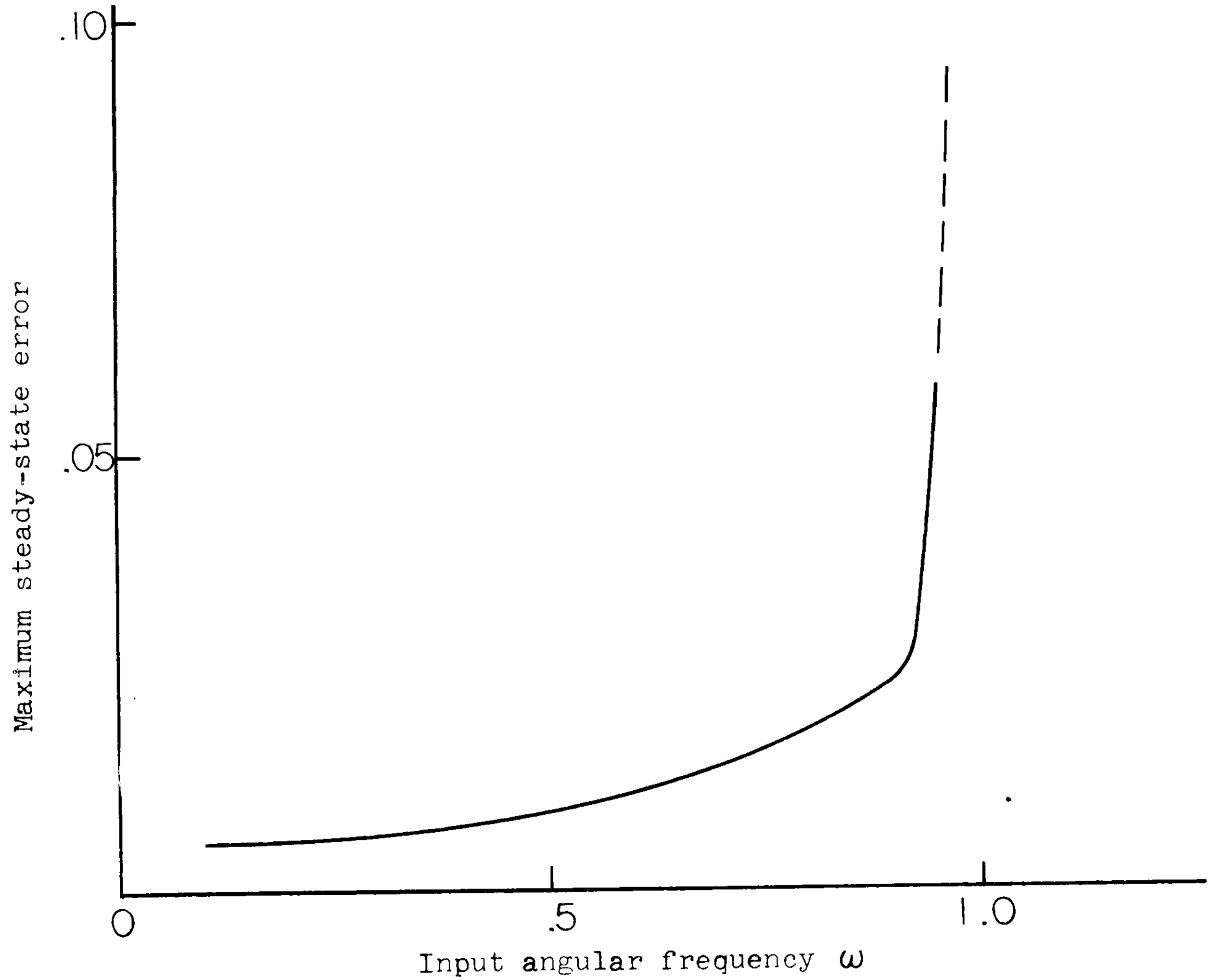


Figure 10.- Concluded.



(a) $\gamma = 0$; $k_1 = 1.00$; $k_2 = 0.37$.

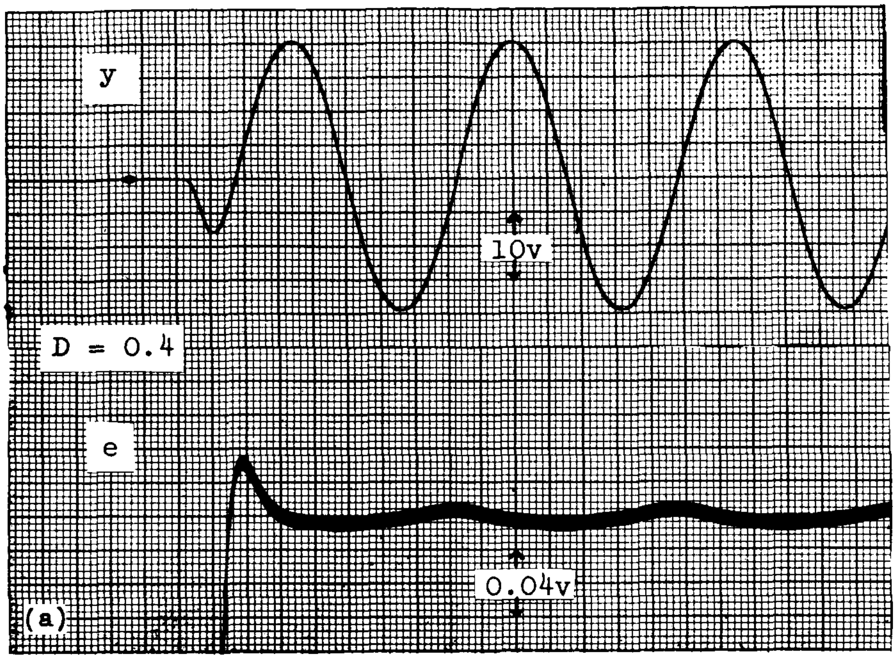
Figure 11.- Steady-state error versus input frequency for a sinusoidal input $x = 20 \cos \omega\tau$. $D = 0.2$; $N = 25$.



(b) $\gamma = 1$; $k_1 = 1.15$; $k_2 = 0.28$.

Figure 11.- Concluded.

W-138



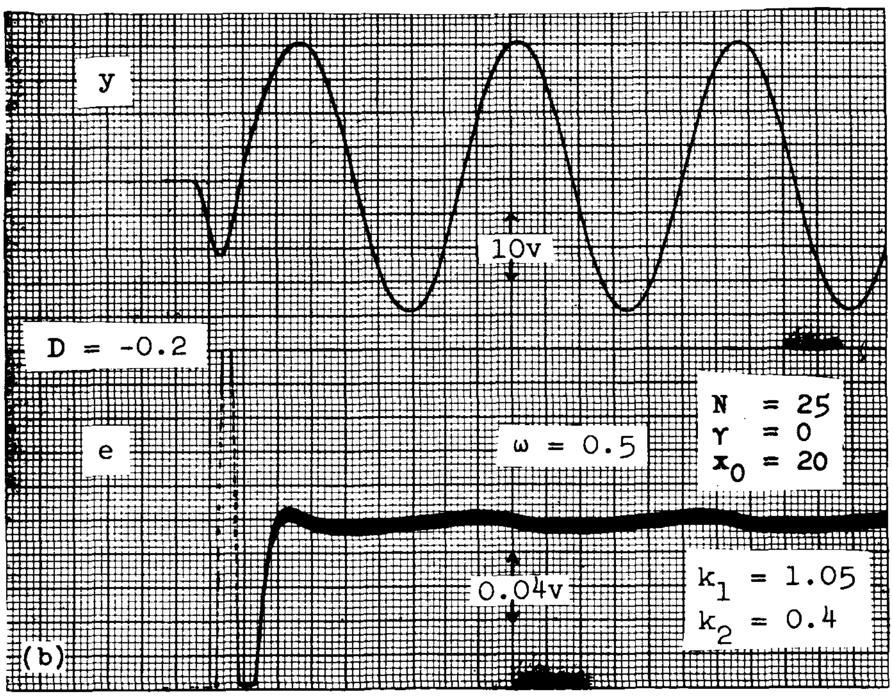


Figure 12.- Response of system to sinusoidal input.

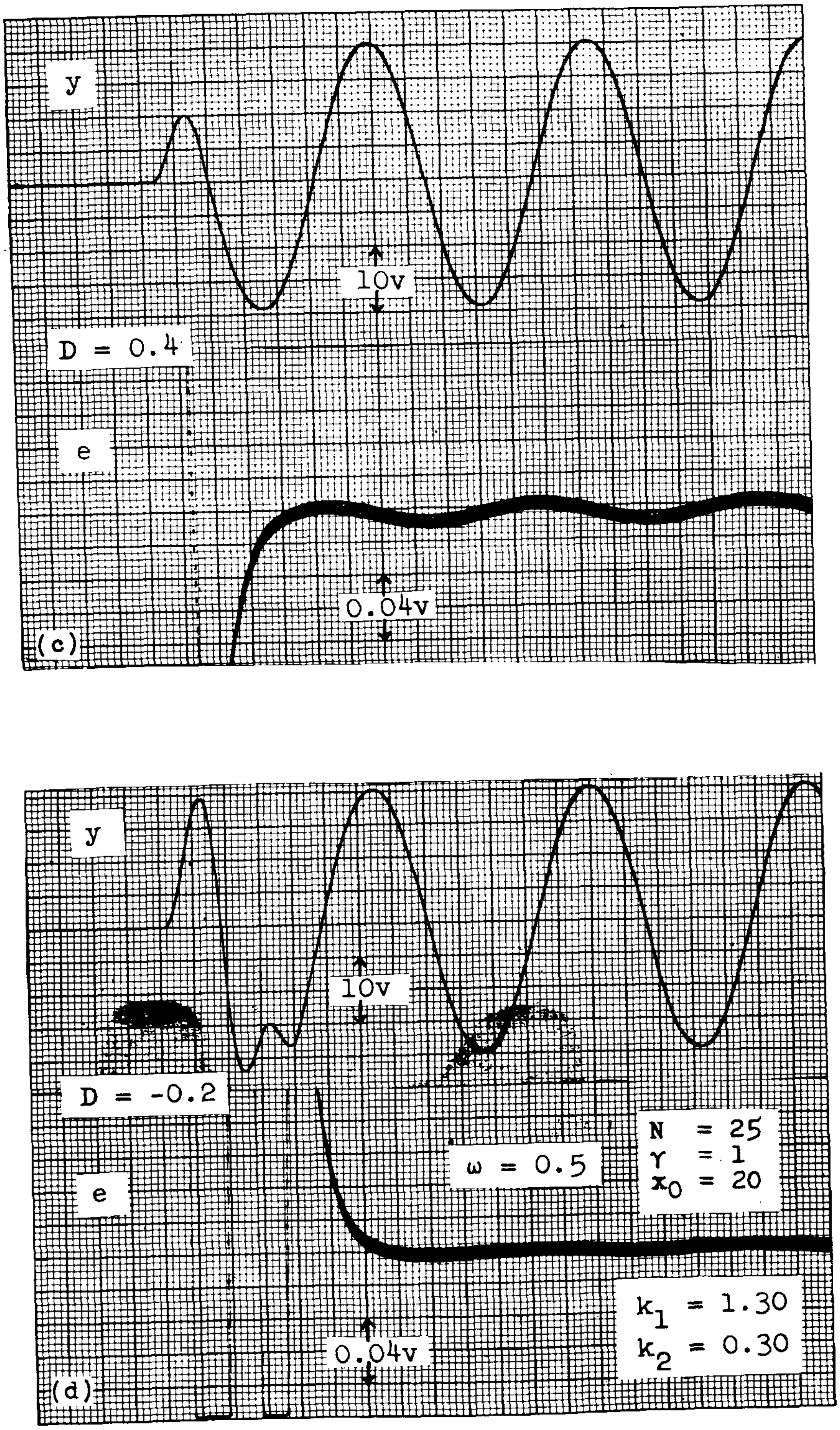
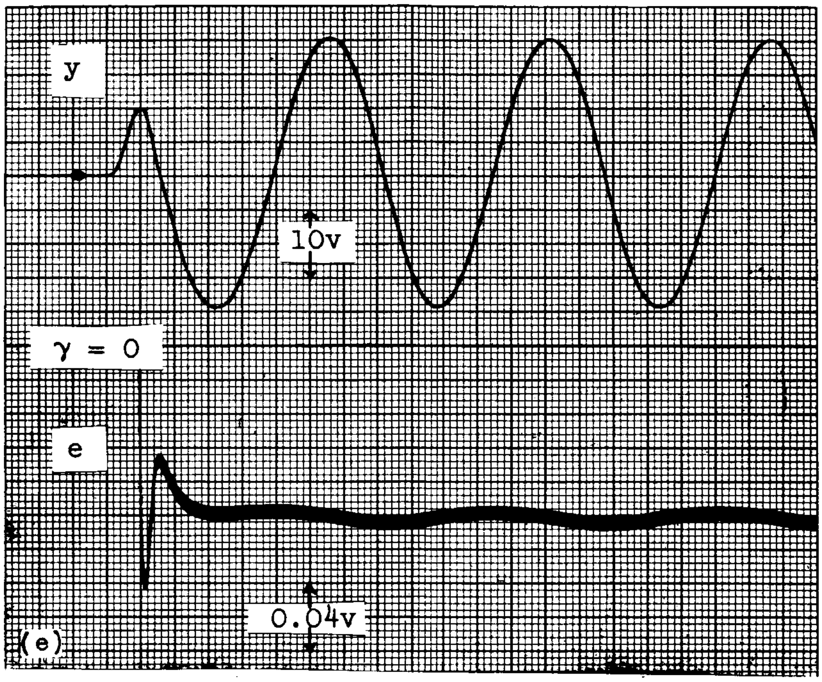


Figure 12.- Continued.



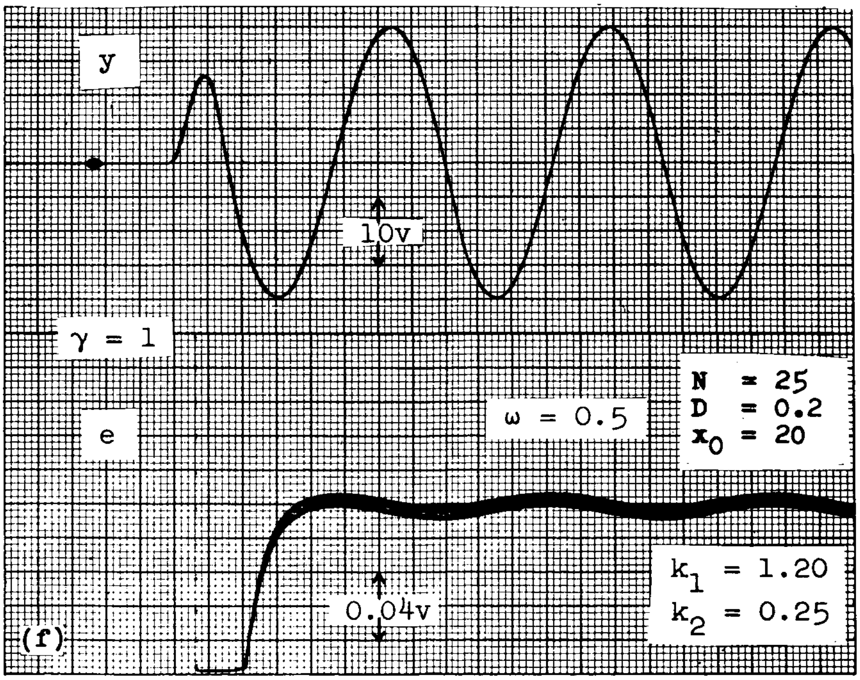


Figure 12.- Continued.

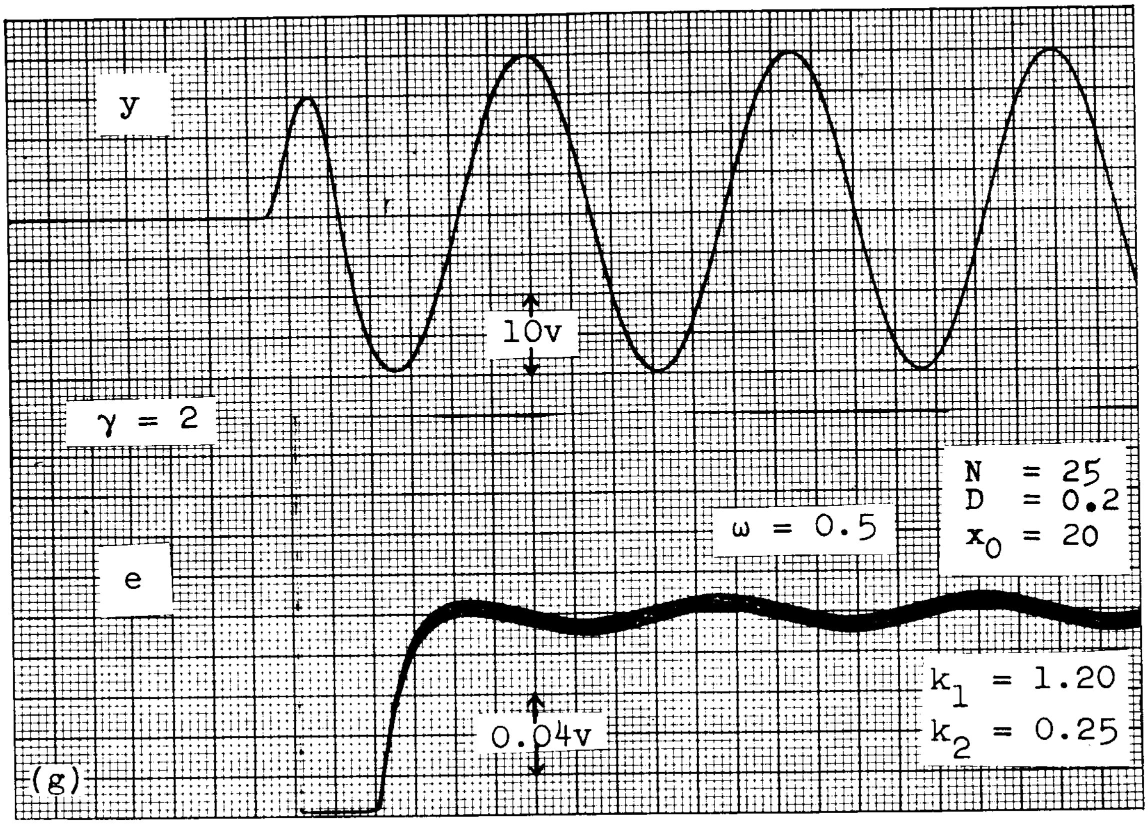


Figure 12.- Concluded.

W-138

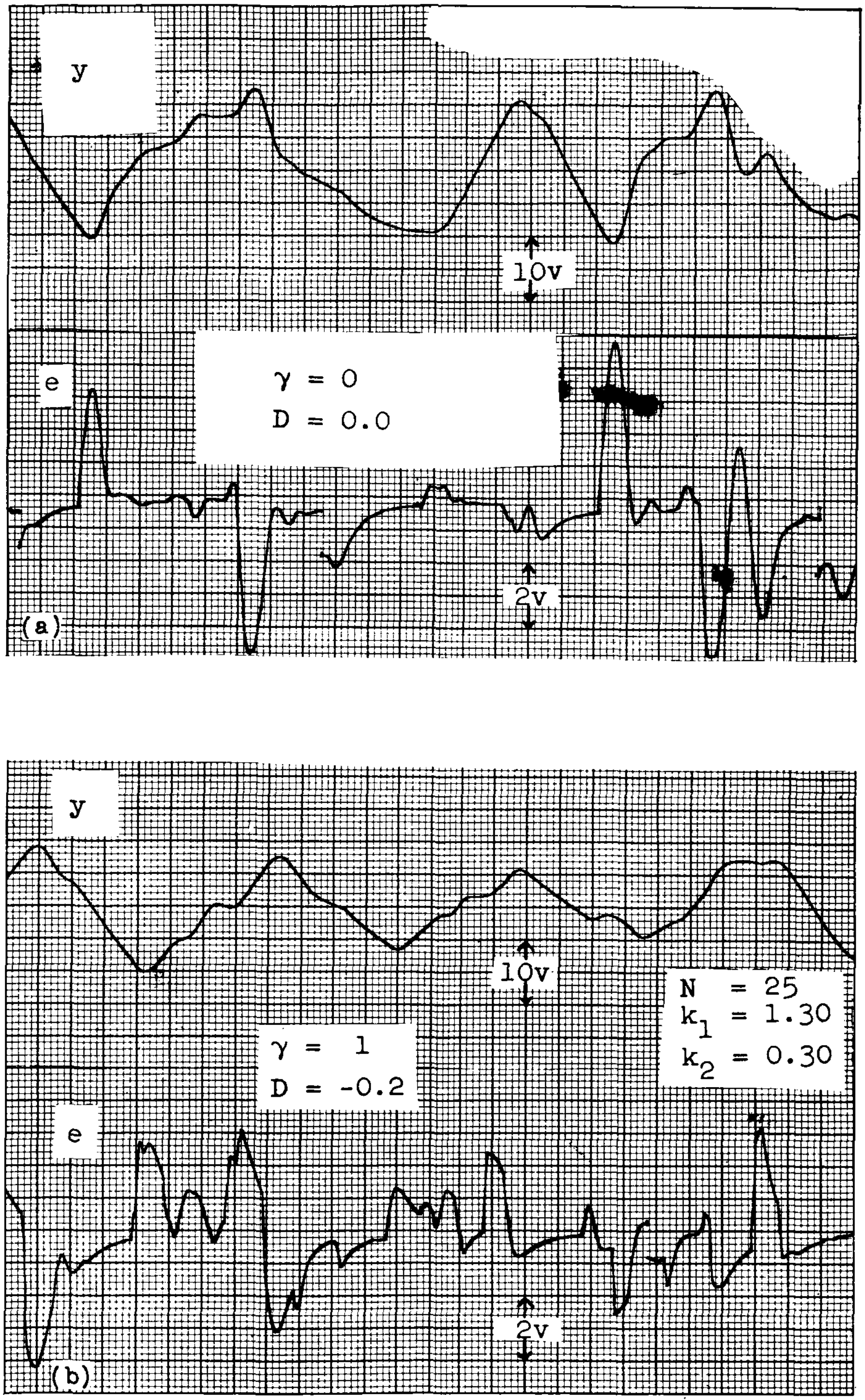


Figure 13.- Response of system to irregular input.

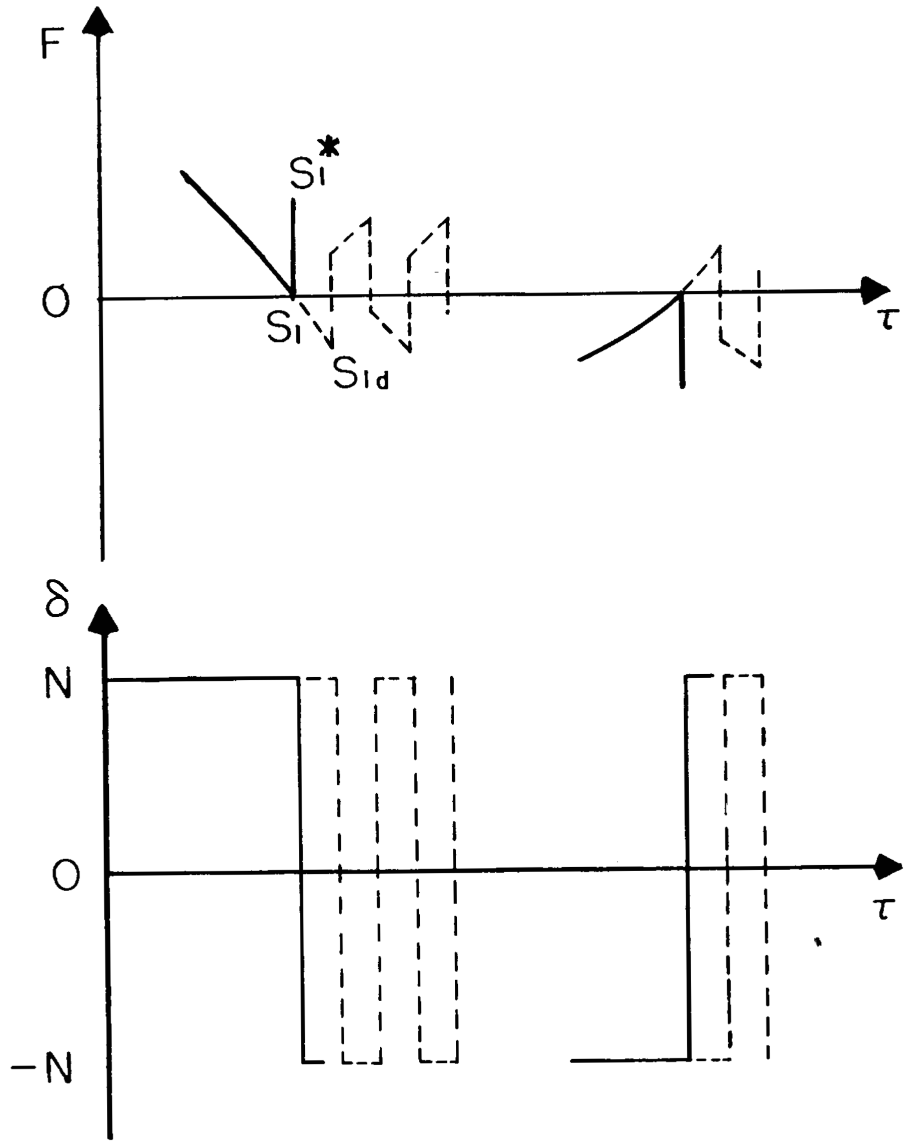


Figure 14.- Behavior of F at switching points. $b_1 > 0$; $b_2 = 0$; $k_2 > 0$; $\Delta_1 F = k_2 \Delta_1 e'' = -b_1 k_2 \Delta_1 \delta$.

W-138

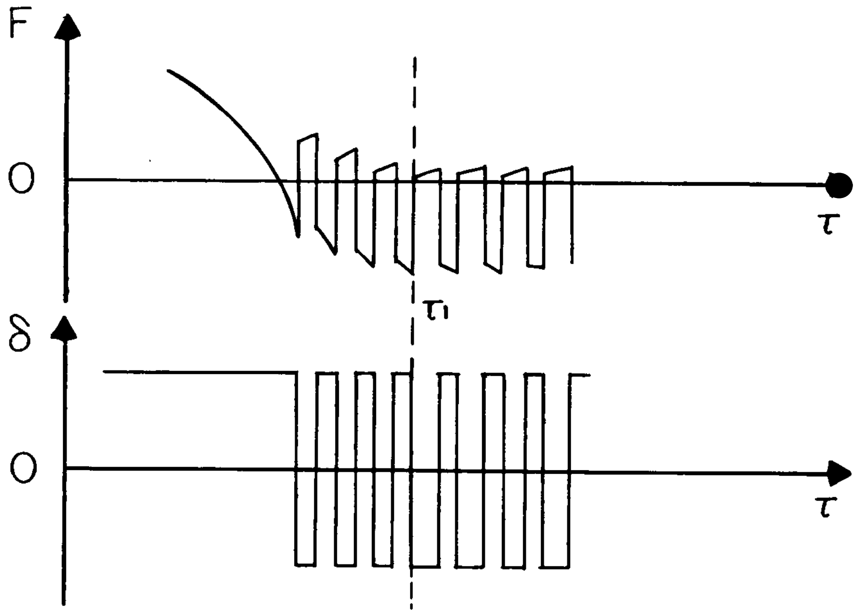


Figure 15.- Sketch of $F(\tau)$ in chatter region showing change from uncontrolled chatter to controlled chatter.

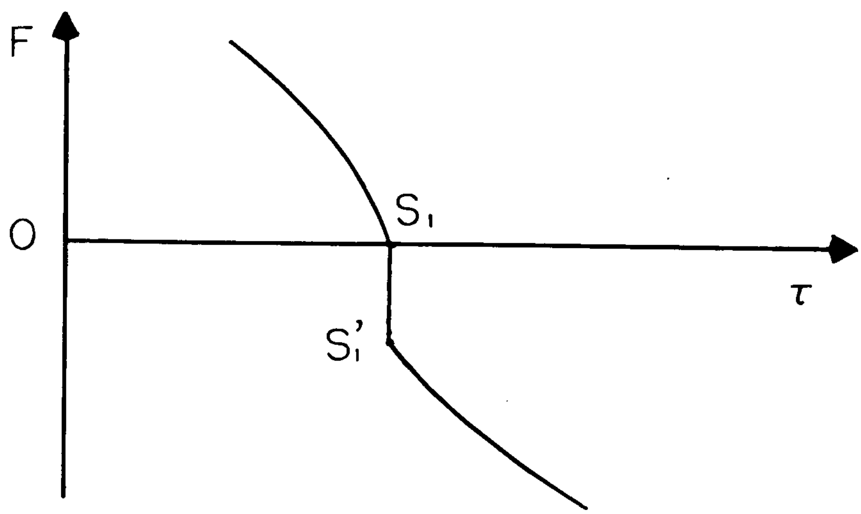
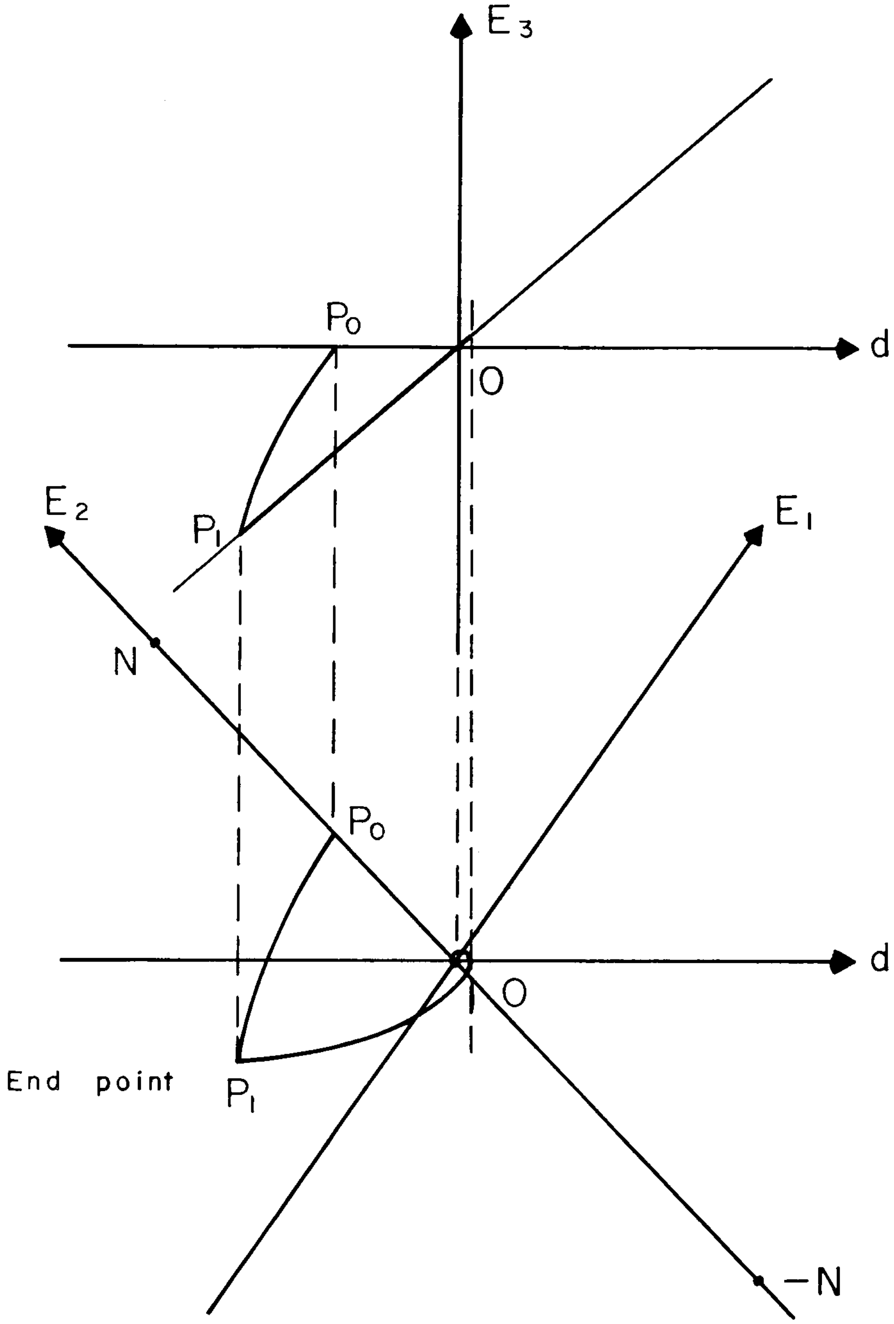
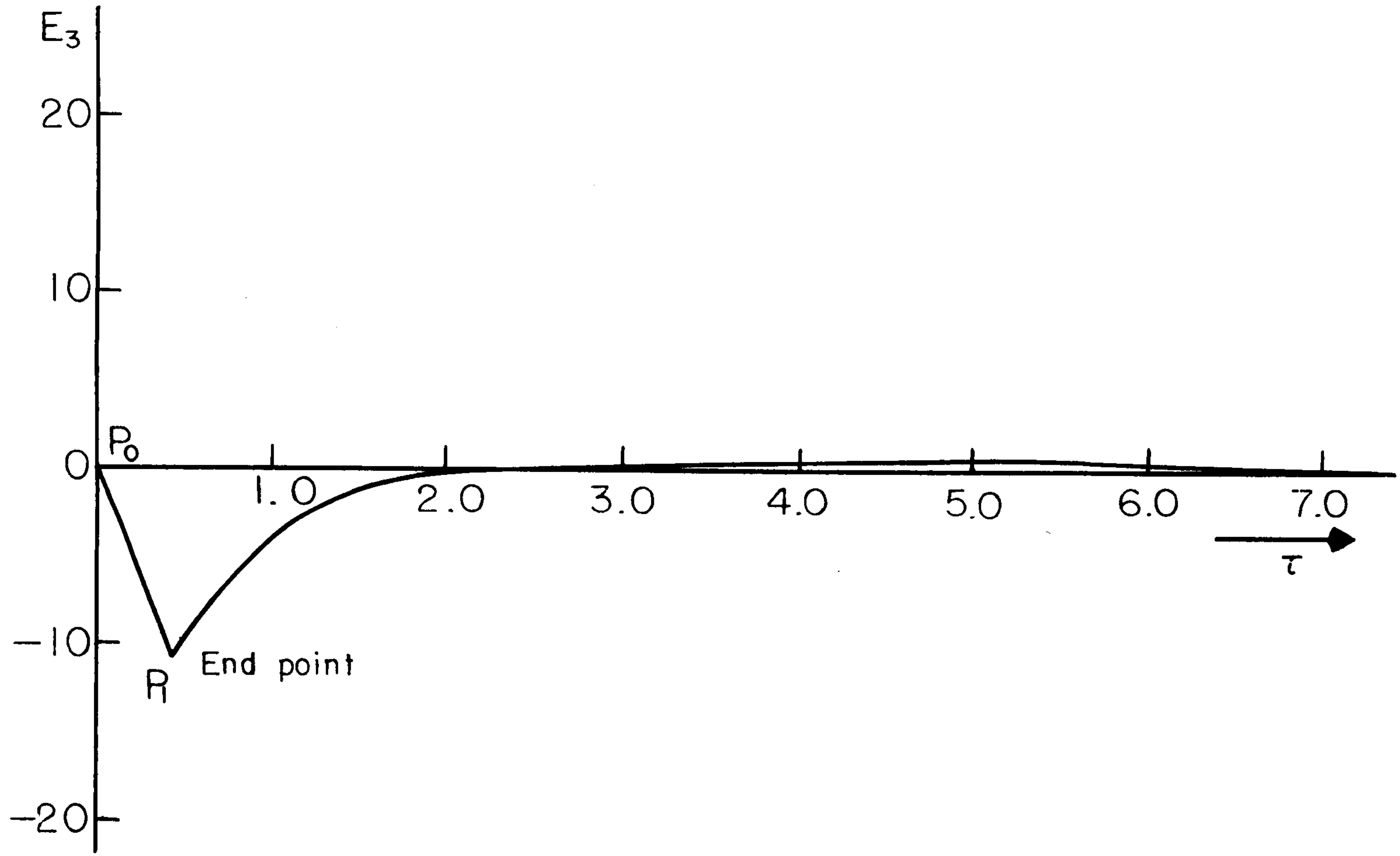


Figure 16.- Sketch of $F(\tau)$ for $b_1 > 0$ and $k_2 < 0$.



(a) Projections into E_1E_2 plane and E_3d plane.

Figure 17.- Illustration of after-end-point motion. N = 25; D = 0.2; $\gamma = 0$; $k_1 = 1.0$; $k_2 = 0.5$.



(b) E_3 against time τ.

Figure 17.- Concluded.